The Principles Underlying Evaluation Estimators with an Application to Matching\textsuperscript{1}

James J. Heckman

The University of Chicago and University College Dublin

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This paper has two objectives. The first is to review the basic principles underlying the identification of conventional econometric evaluation estimators and their recent extensions. The second is to apply the analysis to make explicit the implicit assumptions used in the method of matching. Matching is a version of nonparametric least squares and it suffers from the problem associated with ordinary least squares. It assumes that, fortuitously, the regressors the analyst has at his disposal are “exogenous” in a sense I make precise in this paper. As in the method of least squares, without additional information, there is no way to test if the regressors satisfy the required exogeneity conditions.

The paper is in four parts. The first part discusses general identification assumptions for leading econometric estimators at an intuitive level. The second part presents a prototypical economic choice model. It is a framework that is useful for analyzing and motivating the economic assumptions underlying alternative estimators. The third part discusses the economic assumptions underlying the currently fashionable method of matching. The fourth part concludes.

1 The Basic Principles Underlying the Identification of the Leading Econometric Evaluation Estimators

This section reviews the main principles underlying the evaluation estimators commonly used in the econometric literature. I assume two potential outcomes \( (Y_0, Y_1) \). \( D = 1 \) if \( Y_1 \) is observed, and \( D = 0 \) corresponds to \( Y_0 \) being observed. The observed outcome is

\[
Y = DY_1 + (1 - D)Y_0. \tag{1}
\]

The evaluation problem arises because for each person we observe either \( Y_0 \) or \( Y_1 \) but not both. Thus in general it is not possible to identify the individual level treatment effect \( Y_1 - Y_0 \) for any person. The typical solution to this problem is to reformulate the problem at the population level rather than at the individual level and to identify certain mean outcomes or quantile outcomes or various distributions of outcomes as described in Heckman and Vytlacil (2007a). For example, a commonly used approach focuses attention on average treatment effects, such as
ATE = E(Y_1 - Y_0).

If treatment is assigned or chosen on the basis of potential outcomes, so $$(Y_0, Y_1) \not\perp D,$$

where $\not\perp$ denotes “is not independent” and $\perp$ denotes independent, we encounter the problem of selection bias. Suppose that we observe people in each treatment state $D = 0$ and $D = 1$. If $Y_j \not\perp D$, then the observed $Y_j$ will be selectively different from randomly assigned $Y_j, j = 0, 1$. Thus $E(Y_0 | D = 0) \neq E(Y_0)$ and $E(Y_1 | D = 1) \neq E(Y_1)$. Using unadjusted data to construct $E(Y_1 - Y_0)$ will produce one source of evaluation bias:

$$E(Y_1 | D = 1) - E(Y_0 | D = 0) \neq E(Y_1 - Y_0).$$

The selection problem underlies the evaluation problem. Many methods have been proposed to solve both problems.

The method with the greatest intuitive appeal, which is sometimes called the “gold standard” in evaluation analysis, is the method of random assignment. Nonexperimental methods can be organized by how they attempt to approximate what can be obtained by an ideal random assignment. If treatment is chosen at random with respect to $(Y_0, Y_1)$, or if treatments are randomly assigned and there is full compliance with the treatment assignment,

(R-1) $(Y_0, Y_1) \perp D$.

It is useful to distinguish several cases where (R-1) will be satisfied. The first is that agents (decision makers whose choices are being analyzed) pick outcomes that are random with respect to $(Y_0, Y_1)$. Thus agents may not know $(Y_0, Y_1)$ at the time they make their choices to participate in treatment or at least do not act on $(Y_0, Y_1)$, so that $Pr(D = 1 | X, Y_0, Y_1) = Pr(D = 1 | X)$ for all $X$. Matching assumes a version of (R-1) conditional on matching variables $X$: $(Y_0, Y_1) \perp D | X$.

A second case arises when individuals are randomly assigned to treatment status even if they would choose to self select into no-treatment status, and they comply with the randomization protocols. Let $\xi$ be randomized assignment status. With full compliance, $\xi = 1$ implies that $Y_1$ is
observed and $\xi = 0$ implies that $Y_0$ is observed. Then, under randomized assignment,

$$Y_0, Y_1 \perp \xi,$$

even if in a regime of self-selection, $(Y_0, Y_1) \not\perp D$. If randomization is performed conditional on $X$, we obtain $(Y_0, Y_1) \perp \xi | X$.

Let $A$ denote actual treatment status. If the randomization has full compliance among participants, $\xi = 1 \Rightarrow A = 1; \xi = 0 \Rightarrow A = 0$. This is entirely consistent with a regime in which a person would choose $D = 1$ in the absence of randomization, but would have no treatment ($A = 0$) if suitably randomized, even though the agent might desire treatment.

If treatment status is chosen by self-selection, $D = 1 \Rightarrow A = 1$ and $D = 0 \Rightarrow A = 0$. If there is imperfect compliance with randomization, $\xi = 1 \Rightarrow A = 1$ because of agent choices. In general, $A = \xi D$ so that $A = 1$ only if $\xi = 1$ and $D = 1$. If treatment status is randomly assigned, either through randomization or randomized self-selection,

$$Y_0, Y_1 \perp A.$$

This version of randomization can also be defined conditional on $X$. Under (R-1), (R-2), or (R-3), the average treatment effect (ATE) is the same as the marginal treatment effect of Björklund and Moffitt (1987) and Heckman and Vytlacil (1999, 2005, 2007a), and the parameters treatment on the treated (TT) ($E(Y_1 - Y_0 | D = 1)$) and treatment on the untreated (TUT) ($E(Y_1 - Y_0 | D = 0)$).\footnote{The marginal treatment effect is formally defined in the next section.}

These parameters can be identified from population means:

$$TT = MTE = TUT = ATE = E(Y_1 - Y_0) = E(Y_1) - E(Y_0).$$

Forming averages over populations of persons who are treated ($A = 1$) or untreated ($A = 0$) suffices to identify this parameter. If there are conditioning variables $X$, we can define the mean treatment parameters for all $X$ where (R-1) or (R-2) or (R-3) hold.

Observe that even with random assignment of treatment status and full compliance, one cannot, in general, identify the distribution of the treatment effects ($Y_1 - Y_0$), although one can identify the marginal distributions $F_1(Y_1 | A = 1, X = x) = F_1(Y_1 | X = x)$ and $F_0(Y_0 | A = \xi = 0)$.
One special assumption, common in the conventional econometrics literature, is that \( Y_1 - Y_0 = \Delta(x) \), a constant given \( x \). Since \( \Delta(x) \) can be identified from \( E(Y_1 | A = 1, X = x) - E(Y_0 | A = 0, X = x) \) because \( A \) is allocated by randomization, in this special case the analyst can identify the joint distribution of \((Y_0, Y_1)\). This approach assumes that \((Y_0, Y_1)\) have the same distribution up to a parameter \( \Delta \) (\( Y_0 \) and \( Y_1 \) are perfectly dependent). One can make other assumptions about the dependence across ranks from perfect positive or negative ranking to independence. The joint distribution of \((Y_0, Y_1)\) or of \((Y_1 - Y_0)\) is not identified unless the analyst can pin down the dependence across \((Y_0, Y_1)\). Thus, even with data from a randomized trial one cannot, without further assumptions, identify the proportion of people who benefit from treatment in the sense of gross gain \( (\Pr(Y_1 \geq Y_0)) \). This problem plagues all evaluation methods. Abbring and Heckman (2007) discuss methods for identifying joint distributions of outcomes.

Assumption (R-1) is very strong. In many cases, it is thought that there is selection bias with respect to \( Y_0, Y_1 \), so persons who select into status 1 or 0 are selectively different from randomly sampled persons in the population. The assumption most commonly made to circumvent problems with (R-1) is that even though \( D \) is not random with respect to potential outcomes, the analyst has access to control variables \( X \) that effectively produce a randomization of \( D \) with respect to \((Y_0, Y_1)\) given \( X \). This is the method of matching, which is based on the following conditional independence assumption:

\[
(M-1) \quad (Y_0, Y_1) \perp D | X.
\]

Conditioning on \( X \) randomizes \( D \) with respect to \((Y_0, Y_1)\). (M-1) assumes that any selective sampling of \((Y_0, Y_1)\) can be adjusted by conditioning on observed variables. (R-1) and (M-1) are different assumptions and neither implies the other. In a linear equations model, assumption (M-1) that \( D \) is independent from \((Y_0, Y_1)\) given \( X \) justifies application of least squares on \( D \) to eliminate selection bias in mean outcome parameters. For means, matching is just nonparametric

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\(^3\)Heckman, Smith, and Clements (1997).

regression.\(^5\) In order to be able to compare \(X\)-comparable people in the treatment regime one must assume

\[(M-2) \ 0 < \Pr(D = 1 \mid X = x) < 1.\]

Assumptions (M-1) and (M-2) justify matching. Assumption (M-2) is required for any evaluation estimator that compares treated and untreated persons. It is produced by random assignment if the randomization is conducted for all \(X = x\) and there is full compliance.

Observe that from (M-1) and (M-2), it is possible to identify \(F_1(Y_1 \mid X = x)\) from the observed data \(F_1(Y_1 \mid D = 1, X = x)\) since we observe the left hand side of

\[F_1(Y_1 \mid D = 1, X = x) = F_1(Y_1 \mid X = x) = F_1(Y_1 \mid D = 0, X = x).\]

The first equality is a consequence of conditional independence assumption (M-1). The second equality comes from (M-1) and (M-2). By a similar argument, we observe the left hand side of

\[F_0(Y_0 \mid D = 0, X = x) = F_0(Y_0 \mid X = x) = F_0(Y_0 \mid D = 1, X = x),\]

and the equalities are a consequence of (M-1) and (M-2). Since the pair of outcomes \((Y_0, Y_1)\) is not identified for anyone, as in the case of data from randomized trials, the joint distributions of \((Y_0, Y_1)\) given \(X\) or of \(Y_1 - Y_0\) given \(X\) are not identified without further information. This is a problem that plagues all selection estimators.

From the data on \(Y_1\) given \(X\) and \(D = 1\) and the data on \(Y_0\) given \(X\) and \(D = 0\), since

\[E(Y_1 \mid D = 1, X = x) = E(Y_1 \mid X = x) = E(Y_1 \mid D = 0, X = x)\]
\[\text{and} \ E(Y_0 \mid D = 0, X = x) = E(Y_0 \mid X = x) = E(Y_0 \mid D = 1, X = x)\]

we obtain

\[E(Y_1 - Y_0 \mid X = x) = E(Y_1 - Y_0 \mid D = 1, X = x) = E(Y_1 - Y_0 \mid D = 0, X = x).\]

Effectively, we have a randomization for the subset of the support of \(X\) satisfying (M-2).

At values of \(X\) that fail to satisfy (M-2), there is no variation in \(D\) given \(X\). One can define the

\(^5\)Barnow, Cain, and Goldberger (1980) present one application of matching in a regression setting.
residual variation in \( D \) not accounted for by \( X \) as

\[
E(x) = D - E(D \mid X = x) = D - Pr(D = 1 \mid X = x).
\]

If the variance of \( E(x) \) is zero, it is not possible to construct contrasts in outcomes by treatment status for those \( X \) values and (M-2) is violated. To see the consequences of this violation in a regression setting, use \( Y = Y_0 + D(Y_1 - Y_0) \) and take conditional expectations, under (M-1), to obtain

\[
E(Y \mid X, D) = E(Y_0 \mid X) + D[E(Y_1 - Y_0 \mid X)].^6
\]

If \( \text{Var}(E(x)) > 0 \) for all \( x \) in the support of \( X \), one can use nonparametric least squares to identify \( E(Y_1 - Y_0 \mid X = x) = \text{ATE}(x) \) by regressing \( Y \) on \( D \) and \( X \). The function identified from the coefficient on \( D \) is the average treatment effect.\(^7\) If \( \text{Var}(E(x)) = 0 \), \( \text{ATE}(x) \) is not identified at that \( x \) value because there is no variation in \( D \) that is not fully explained by \( X \). A special case of matching is linear least squares where one can write

\[
Y_0 = X\alpha + U \quad Y_1 = X\alpha + \beta + U
\]

\( U_0 = U_1 = U \) and hence under (M-1),

\[
E(Y \mid X, D) = X\alpha + \beta D.
\]

If \( D \) is perfectly predictable by \( X \), one cannot identify \( \beta \) because of a multicollinearity problem. (M-2) rules out perfect collinearity.\(^8\) Matching is a nonparametric version of least squares that does not impose functional form assumptions on outcome equations, and that imposes support condition (M-2).

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^6This follows because \( E(Y \mid X, D) = E(Y_0 \mid X, D) + D E(Y_1 - Y_0 \mid X, D) \) but from (M-1), \( E(Y_0 \mid X, D) = E(Y_0 \mid X) \) and \( E(Y_1 - Y_0 \mid X, D) = E(Y_1 - Y_0 \mid X) \).

^7Under the conditional independence assumption (M-1), it is also the effect of treatment on the treated \( E(Y_1 - Y_0 \mid X, D = 1) \) and the marginal treatment effect formally defined in the next section.

^8Clearly (M-1) and (M-2) are sufficient but not necessary conditions. For the special case of OLS, as a consequence of the assumed linearity in the functional form of the estimating equation, we achieve identification of \( \beta \) if \( \text{Cov}(X, U) = 0, \text{Cov}(D, U) = 0 \) and \( (D, X) \) are not perfectly collinear. These conditions are much weaker than (M-1) and (M-2) and can be satisfied if (M-1) and (M-2) are only identified in a subset of the support of \( X \).
Conventional econometric choice models make a distinction between variables that appear in outcome equations \((X)\) and variables that appear in choice equations \((Z)\). The same variables may be in \((X)\) and \((Z)\) but more typically, there are some variables not in common. For example, the instrumental variable estimator is based on variables that are not in \(X\) but that are in \(Z\). Matching makes no distinction between the \(X\) and the \(Z\).\(^9\) It does not rely on exclusion restrictions. The conditioning variables used to achieve conditional independence can in principle be a set of variables \(Q\) distinct from the \(X\) variables (covariates for outcomes) or the \(Z\) variables (covariates for choices). I use \(X\) solely to simplify the notation. The key identifying assumption is the assumed existence of a random variable \(X\) with the properties satisfying (M-1) and (M-2).

Conditioning on a larger vector \((X\) augmented with additional variables) or a smaller vector \((X\) with some components removed) may or may not produce suitably modified versions of (M-1) and (M-2). Without invoking further assumptions there is no objective principle for determining what conditioning variables produce (M-1).

Assumption (M-1) is strong. Many economists do not have enough faith in their data to invoke it. Assumption (M-2) is testable and requires no act of faith. To justify (M-1), it is necessary to appeal to the quality of the data.

Using economic theory can help guide the choice of an evaluation estimator. A crucial distinction is the one between the information available to the analyst and the information available to the agent whose outcomes are being studied. Assumptions made about these information sets drive the properties of econometric estimators. Analysts using matching make strong informational assumptions in terms of the data available to them. In fact, all econometric estimators make assumptions about the presence or absence of informational asymmetries, and I exposit them in this paper.

To analyze the informational assumptions invoked in matching, and other econometric evaluation strategies, it is helpful to introduce five distinct information sets and establish some relationships among them.\(^{10}\) (1) An information set \(\sigma(I_k)\) with an associated random variable

\(^9\)Heckman, Ichimura, Smith, and Todd (1998) distinguish \(X\) and \(Z\) in matching. They consider a case where conditioning on \(X\) may lead to failure of (M-1) and (M-2) but conditioning on \((X,Z)\) satisfies a suitably modified version of this condition.

\(^{10}\)See also the discussion in Barros (1987), Heckman and Navarro (2004), and Gerfin and Lechner (2002).
that satisfies conditional independence (M-1) is defined as a relevant information set; (2) The minimal information set \( \sigma(I_R) \) with associated random variable needed to satisfy conditional independence (M-1), the minimal relevant information set; (3) The information set \( \sigma(I_A) \) available to the agent at the time decisions to participate are made; (4) The information available to the economist, \( \sigma(I_E) \); and (5) The information \( \sigma(I_E) \) used by the economist in conducting an empirical analysis. I will denote the random variables generated by these sets as \( I_{R^*}, I_R, I_A, I_{E^*}, I_E \), respectively.\(^{11}\)

**Definition 1.** Define \( \sigma(I_{R^*}) \) as a relevant information set if the information set is generated by the random variable \( I_{R^*} \), possibly vector valued, and satisfies condition (M-1), so

\[
(Y_0, Y_1) \perp D \mid I_{R^*}.
\]

**Definition 2.** Define \( \sigma(I_R) \) as a minimal relevant information set if it is the intersection of all sets \( \sigma(I_{R^*}) \) and satisfies \( (Y_0, Y_1) \perp D \mid I_R \). The associated random variable \( I_R \) is a minimum amount of information that guarantees that condition (M-1) is satisfied. There may be no such set.\(^{12}\)

If one defines the relevant information set as one that produces conditional independence, it may not be unique. If the set \( \sigma(I_{R^*}) \) satisfies the conditional independence condition, then the set \( \sigma(I_{R^*}, Q) \) such that \( Q \perp (Y_0, Y_1) \mid I_{R^*} \) would also guarantee conditional independence. For this reason, I define the relevant information set to be minimal, that is, to be the intersection of all relevant sets that still produce conditional independence between \( (Y_0, Y_1) \) and \( D \). However, no minimal set may exist.

**Definition 3.** The agent’s information set, \( \sigma(I_A) \), is defined by the information \( I_A \) used by the agent when choosing among treatments. Accordingly, I call \( I_A \) the agent’s information.

\(^{11}\)I start with a primitive probability space \((\Omega, \sigma, P)\) with associated random variables \( I \). I assume minimal \( \sigma \)-algebras and assume that the random variables \( I \) are measurable with respect to these \( \sigma \)-algebras. Obviously, strictly monotonic or affine transformations of the \( I \) preserve the information and can substitute for the \( I \).

\(^{12}\)Observe that the intersection of all sets \( \sigma(I_{R^*}) \) may be empty and hence may not be characterized by a (possibly vector valued) random variable \( I_R \) that guarantees \( (Y_1, Y_2) \perp D \mid I_R \). If the information sets that produce conditional independence are nested, then the intersection of all sets \( \sigma(I_{R^*}) \) producing conditional independence is well defined and has an associated random variable \( I_R \) with the required property, although it may not be unique (e.g., strictly monotonic transformations and affine transformations of \( I_R \) also preserve the property). In the more general case of non-nested information sets with the required property, it is possible that no uniquely defined minimal relevant set exists. Among collections of nested sets that possess the required property, there is a minimal set defined by intersection but there may be multiple minimal sets corresponding to each collection.
By the agent I mean the person making the treatment decision not necessarily the person whose outcomes are being studied (e.g. the agent may be the parent; the person being studied may be a child).

**Definition 4.** The econometrician’s full information set, $\sigma(I_E^*)$, is defined as all of the information available to the econometrician, $I_E^*$.

**Definition 5.** The econometrician’s information set, $\sigma(I_E)$, is defined by the information used by the econometrician when analyzing the agent’s choice of treatment, $I_E$, in conducting an analysis.

For the case where a unique minimal relevant information set exists, only three restrictions are implied by the structure of these sets: $\sigma(I_R) \subseteq \sigma(I_R^*)$, $\sigma(I_R) \subseteq \sigma(I_A)$, and $\sigma(I_E) \subseteq \sigma(I_E^*)$. I have already discussed the first restriction. The second restriction requires that the minimal relevant information set must be part of the information the agent uses when deciding which treatment to take or assign. It is the information in $\sigma(I_A)$ that gives rise to the selection problem which in turn gives rise to the evaluation problem.

The third restriction requires that the information used by the econometrician must be part of the information that the agent observes. Aside from these orderings, the econometrician’s information set may be different from the agent’s or the relevant information set. The econometrician may know something the agent doesn’t know, for typically he is observing events after the decision is made. At the same time, there may be private information known to the agent but not the econometrician. Matching assumption (M-1) implies that $\sigma(I_R) \subseteq \sigma(I_E)$, so that the econometrician uses at least the minimal relevant information set, but of course he or she may use more. However, using more information is not guaranteed to produce a model with conditional independence property (M-1) satisfied for the augmented model. Thus an analyst can “overdo” it. I present examples of the consequences of the asymmetry in agent and analyst information sets in section 3.

The possibility of asymmetry in information between the agent making participation decisions and the observing economist creates the potential for a major identification problem that

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13This formulation assumes that the agent makes the treatment decision. The extension to the case where the decision maker and the agent are distinct is straightforward. The requirement $\sigma(I_R) \subseteq \sigma(I_R^*)$ is satisfied by nested sets.
is ruled out by assumption (M-1). The methods of control functions and instrumental variables estimators (and closely related regression discontinuity design methods) address this problem but in different ways. Accounting for this possibility is a more conservative approach to the selection problem than the one taken by advocates of least squares, or its nonparametric counterpart, matching. Those advocates assume that they know the $X$ that produces a relevant information set. Below, I show the biases that can result in matching when standard econometric model selection criteria are applied to pick the $X$ that are used to satisfy (M-1). Conditional independence condition (M-1) cannot be tested without maintaining other assumptions.\textsuperscript{14} Choice of the appropriate conditioning variables is a problem that plagues all econometric estimators.

The methods of control functions, replacement functions, proxy variables, and instrumental variables all recognize the possibility of asymmetry in information between the agent being studied and the econometrician and recognize that even after conditioning on $X$ (variables in the outcome equation) and $Z$ (variables affecting treatment choices, which may include the $X$), analysts may fail to satisfy conditional independence condition (M-1).\textsuperscript{15} These methods postulate the existence of some unobservables $\theta$, which may be vector valued, with the property that

\[(U-1) \ (Y_0, Y_1) \perp D \mid X, Z, \theta,\]

but allow for the possibility that

\[(U-2) \ (Y_0, Y_1) \not\perp D \mid X, Z.\]

In the event (U-2) holds, these approaches model the relationships of the unobservable $\theta$ with $Y_1, Y_0$ and $D$ in various ways. The content in the control function principle is to specify the exact nature of the dependence on the relationship between observables and unobservables in a nontrivial fashion that is consistent with economic theory. I present examples of models that satisfy (U-1) but not (U-2) in section 3.

\textsuperscript{14}These assumptions may or may not be testable. I discuss the required “exogeneity” conditions below in my discussion of matching in section 3. Thus randomization of assignment of treatment status might be used to test (M-1) but this requires that there be full compliance and that the randomization be valid (no anticipation effects or general equilibrium effects).

\textsuperscript{15}The term and concept of control function is due to Heckman and Robb (1985a,b, 1986a,b). See Blundell and Powell (2003) (who call the Heckman-Robb replacement functions control functions). A more recent nomenclature is “control variate”. Matzkin (2007) provides a comprehensive discussion of identification principles for econometric estimators.
The early literature focused on mean outcomes conditional on covariates (Heckman and Robb, 1985a,b, 1986a,b) and assumes a weaker version of (U-1) based on conditional mean independence rather than full conditional independence. More recent work analyzes distributions of outcomes (e.g. Aakvik, Heckman, and Vytlacil, 2005; Carneiro, Hansen, and Heckman, 2001, 2003). This work is reviewed in Abbring and Heckman (2007).

The normal Roy selection model makes distributional assumptions and identifies the joint distribution of outcomes. A large literature surveyed by Matzkin (2007) makes alternative assumptions to satisfy (U-1) in nonparametric settings. Replacement functions (Heckman and Robb, 1985a) are methods that proxy $\theta$. They substitute out for $\theta$ using observables.\footnote{This is the “control variate” of Blundell and Powell (2003). Heckman and Robb (1985a) and Olley and Pakes (1996) use a similar idea. Matzkin (2007) discusses replacement functions.} Aakvik, Heckman, and Vytlacil (1999, 2005), Carneiro, Hansen, and Heckman (2001, 2003), Cunha, Heckman, and Navarro (2005), and Cunha, Heckman, and Schennach (2006a,b) develop methods that integrate out $\theta$ from the model assuming $\theta \perp (X,Z)$, or invoking weaker mean independence assumptions, and assuming access to proxy measurements for $\theta$. They also consider methods for estimating the distributions of treatment effects. These are discussed in Abbring and Heckman (2007).

The normal selection model produces partial identification of a generalized Roy model and full identification of a Roy model under separability and normality. It models the conditional expectation of $U_0$ and $U_1$ given $X,Z,D$. In terms of (U-1), it models the conditional mean dependence of $Y_0, Y_1$ on $D$ and $\theta$ given $X$ and $Z$. Powell (1994) and Matzkin (2007) survey methods for producing semiparametric versions of these models. Heckman and Vytlacil (2007a, Appendix B) or the appendix of Heckman and Navarro (2007) present a prototypical identification proof for a general selection model that implements (U-1) by estimating the distribution of $\theta$, assuming $\theta \perp (X,Z)$, and invoking support conditions on $(X,Z)$.

Central to both the selection approach and the instrumental variable approach for a model with heterogenous responses is the probability of selection. Let $Z$ denote variables in the choice equation. Fixing $Z$ at different values (denoted $z$) I define $D(z)$ as an indicator function that is “1” when treatment is selected at the fixed value of $z$ and that is “0” otherwise. In terms of a
separable index model $U_D = \mu_D(Z) - V$, for a fixed value of $z$,

$$D(z) = 1[\mu_D(z) \geq V]$$

where $Z \perp V | X$. Thus fixing $Z = z$, values of $z$ do not affect the realizations of $V$ for any value of $X$. An alternative way of representing the independence between $Z$ and $V$ given $X$ due to Imbens and Angrist (1994), writes that $D(z) \perp Z$ for all $z \in Z$, where $Z$ is the support of $Z$. The Imbens-Angrist independence condition for IV is

$$\{D(z)\}_{z \in Z} \perp Z | X.$$ 

Thus the probabilities that $D(z) = 1, z \in Z$ are not affected by the occurrence of $Z$. Vytlacil (2002) establishes the equivalence of these two formulations under general conditions.\(^\text{17}\)

The method of instrumental variables (IV) postulates that

(IV-1) \( (Y_0, Y_1, \{D(z)\}_{z \in Z}) \perp Z | X. \) (Independence)

One consequence of this assumption is that $E(D | Z) = P(Z)$, the propensity score, is random with respect to potential outcomes. Thus $(Y_0, Y_1) \perp P(Z) | X$. So are all other functions of $Z$ given $X$. The method of instrumental variables also assumes that

(IV-2) $E(D | X, Z) = P(X, Z)$ is a nondegenerate function of $Z$ given $X$. (Rank Condition)

Alternatively, one can write that $\text{Var}(E(D | X, Z)) \neq \text{Var}(E(D | X))$.

Comparing (IV-1) to (M-1) in the method of instrumental variables, $Z$ is independent of $(Y_0, Y_1)$ given $X$ whereas in matching $D$ is independent of $(Y_0, Y_1)$ given $X$. So in (IV-1), $Z$ plays the role of $D$ in matching condition (M-1). Comparing (IV-2) with (M-2), in the method of IV the choice probability $\Pr(D = 1 | X, Z)$ is assumed to vary with $Z$ conditional on $X$, whereas in matching, $D$ varies conditional on $X$. Unlike the method of control functions, no explicit model of the relationship between $D$ and $(Y_0, Y_1)$ is required in applying IV.

\(^{17}\)See Heckman and Vytlacil (2007b) for a discussion of these conditions.
(IV-2) is a rank condition and can be empirically verified. (IV-1) is not testable as it involves assumptions about counterfactuals. In a conventional common coefficient regression model

\[ Y = \alpha + \beta D + U, \]

where \( \beta \) is a constant and where I allow for \( \text{Cov}(D, U) \neq 0 \), (IV-1) and (IV-2) identify \( \beta \).\(^{18}\) When \( \beta \) varies in the population and is correlated with \( D \), additional assumptions must be invoked for IV to identify interpretable parameters. Heckman, Urzua, and Vytlacil (2006) and Heckman and Vytlacil (2007b) discuss these conditions.

Assumptions (IV-1) and (IV-2), with additional assumptions in the case where \( \beta \) varies in the population which I discuss in this paper, can be used to identify mean treatment parameters. Replacing \( Y_1 \) with \( 1(Y_1 \leq t) \) and \( Y_0 \) with \( 1(Y_0 \leq t) \), where \( t \) is a constant, the IV approach allows us to identify marginal distributions \( F_1(y_1 | X) \) or \( F_0(y_0 | X) \).

In matching, the variation in \( D \) that arises after conditioning on \( X \) provides the source of randomness that switches people across treatment status. Nature is assumed to provide an experimental manipulation conditional on \( X \) that replaces the randomization assumed in (R-1)-(R-3). When \( D \) is perfectly predictable by \( X \), there is no variation in it conditional on \( X \), and the randomization by nature breaks down. Heuristically, matching assumes a residual \( \mathcal{E}(X) = D - E(D | X) \) that is nondegenerate and is one manifestation of the randomness that causes persons to switch status.\(^{19}\)

In the IV method, it is the choice probability \( E(D | X, Z) = P(X, Z) \) that is random with respect to \( (Y_0, Y_1) \), not components of \( D \) not predictable by \( (X, Z) \). Variation in \( Z \) for a fixed \( X \) provides the required variation in \( D \) that switches treatment status and still produces the required conditional independence:

\[ (Y_0, Y_1) \perp P(X, Z) | X. \]

Variation in \( P(X, Z) \) produces variations in \( D \) that switch treatment status. Components of variation in \( D \) not predictable by \( (X, Z) \) do not produce the required independence. Instead, the

\[ \hat{\beta} = \frac{\text{Cov}(Z, Y)}{\text{Cov}(Z, D)}. \]

\(^{18}\)It is heuristically illuminating, but technically incorrect to replace \( \mathcal{E}(X) \) with \( D \) in (R-1) or \( R \) in (R-2) or \( T \) in (R-3). In general \( \mathcal{E}(X) \) is not independent of \( X \) even if it is mean independent.
predicted component provides the required independence. It is just the opposite in matching. Versions of the method of control functions use measurements to proxy $\theta$ in (U-1) and (U-2) and remove spurious dependence that gives rise to selection problems. These are called replacement functions (see Heckman and Robb, 1985a) or control variates (see Blundell and Powell, 2003).

The methods of replacement functions and proxy variables all start from characterizations (U-1) and (U-2). $\theta$ is not observed and $(Y_0, Y_1)$ are not observed directly but $Y$ is observed:

$$Y = DY_1 + (1 - D) Y_0.$$  

Missing variables $\theta$ produce selection bias which creates a problem with using observational data to evaluate social programs. From (U-1), if one conditions on $\theta$, condition (M-1) for matching would be satisfied, and hence one could identify the parameters and distributions that can be identified if the conditions required for matching are satisfied.

The most direct approach to controlling for $\theta$ is to assume access to a function $\tau(X, Z, Q)$ that perfectly proxies $\theta$:

$$\theta = \tau(X, Z, Q). \quad (2)$$

This approach based on a perfect proxy is called the method of replacement functions by Heckman and Robb (1985a). In (U-1), one can substitute for $\theta$ in terms of observables $(X, Z, Q)$. Then

$$(Y_0, Y_1) \perp D \mid X, Z, Q.$$  

It is possible to condition nonparametrically on $(X, Z, Q)$ and without having to know the exact functional form of $\tau$. $\theta$ can be a vector and $\tau$ can be a vector of functions. This method has been used in the economics of education for decades (see the references in Heckman and Robb, 1985a). If $\theta$ is ability and $\tau$ is a test score, it is sometimes assumed that the test score is a perfect proxy (or replacement function) for $\theta$ and that one can enter it into the regressions of earnings on schooling to escape the problem of ability bias (typically assuming a linear relationship between
\(\tau\) and \(\theta\).\(^{20}\) Heckman and Robb (1985a) discuss the literature that uses replacement functions in this way. Olley and Pakes (1996) apply this method and consider nonparametric identification of the \(\tau\) function. Matzkin (2007) provides a rigorous proof of identification for this approach in a general nonparametric setting.

The method of replacement functions assumes that (2) is a perfect proxy. In many applications, this assumption is far too strong. More often, \(\theta\) is measured with error. This produces a factor model or measurement error model (Aigner, Hsiao, Kapteyn, and Wansbeek, 1984). Matzkin (2007) surveys this method. One can represent the factor model in a general way by a system of equations:

\[
Y_j = g_j(X, Z, Q, \theta, \varepsilon_j), \quad j = 1, \ldots, J. \tag{3}
\]

A linear factor model separable in the unobservables writes

\[
Y_j = g_j(X, Z, Q) + \alpha_j\theta + \varepsilon_j, \quad j = 1, \ldots, J, \tag{4}
\]

where

\[
(X, Z, Q) \perp (\theta, \varepsilon_j), \varepsilon_j \perp \theta, \quad j = 1, \ldots, J, \tag{5}
\]

and the \(\varepsilon_j\) are mutually independent. Observe that under (3) and (4), \(Y_j\) controlling for \(X, Z, Q\) only imperfectly proxies \(\theta\) because of the presence of \(\varepsilon_j\). \(\theta\) is called a factor, \(\alpha_j\) factor loadings and the \(\varepsilon_j\) “uniquenesses” (see, e.g. Aigner, 1985).

A large literature, reviewed in Abbring and Heckman (2007) and Matzkin (2007) shows how to establish identification of econometric models under factor structure assumptions. Cunha, Heckman, and Matzkin (2003), Schennach (2004) and Hu and Schennach (2006) establish identification in nonlinear models of the form (3).\(^{21}\) The key to identification is multiple, but imperfect (because of \(\varepsilon_j\)), measurements on \(\theta\) from the \(Y_j, j = 1, \ldots, J\) and \(X, Z, Q\), and possibly other

\(^{20}\)Thus if \(\tau = \alpha_0 + \alpha_1X + \alpha_2Q + \alpha_3Z + \theta\), one can write

\[
\theta = \tau - \alpha_0 - \alpha_1X - \alpha_2Q - \alpha_3Z,
\]

and use this as the proxy function. Controlling for \(T, X, Q, Z\) controls for \(\theta\). Notice that one does not need to know the coefficients \((\alpha_0, \alpha_1, \alpha_2, \alpha_3)\) to implement the method. one can condition on \(X, Q, Z\).

\(^{21}\)Cunha, Heckman, and Schennach (2006a,b) apply and extend this approach to a dynamic factor setting where the \(\theta_j\) are time dependent.
measurement systems that depend on $\theta$. Carneiro, Hansen, and Heckman (2003), Cunha, Heckman, and Navarro (2005, 2006) and Cunha and Heckman (2006a,b) apply and develop these methods. Under assumption (5), they show how to nonparametrically identify the econometric model and the distributions of the unobservables $F_\theta(\theta)$ and $F_{\varepsilon_j}(\varepsilon_j)$. In the context of classical simultaneous equations models, identification is secured by using covariance restrictions across equations exploiting the low dimensionality of vector $\theta$ compared to the high dimensional vector of (imperfect) measurements on it. The recent literature (Cunha, Heckman, and Matzkin, 2003; Cunha, Heckman, and Schennach, 2006b; Hu and Schennach, 2006) extends the linear model to a nonlinear setting.

The recent econometric literature applies in special cases the idea of the control function principle introduced in Heckman and Robb (1985a). This principle, versions of which can be traced back to Telser (1964), partitions $\theta$ in (U-1) into two or more components, $\theta = (\theta_1, \theta_2)$, where only one component of $\theta$ is the source of bias. Thus it is assumed that (U-1) is true, and (U-1)$'$ is also true:

$$(U-1)' \quad (Y_0, Y_1) \perp D \mid X, Z, \theta_1,$$

and (U-2) holds. For example, in a normal selection model with additive separability, one can break $U_1$, the error term associated with $Y_1$, into two components:

$$U_1 = E(U_1 \mid V) + \varepsilon,$$

where $V$ plays the role of $\theta_1$ and is associated with the choice equation. Under normality, $\varepsilon$ is independent of $E(U_1 \mid V)$. Further,

$$E(U_1 \mid V) = \frac{\text{Cov}(U_1, V)}{\text{Var}(V)}V,$$  \hspace{1cm} (6)

assuming $E(U_1) = 0$ and $E(V) = 0$. Heckman and Robb (1985a) show how to construct a control function in the context of the choice model

$$D = 1[\mu_D(Z) > V].$$
Controlling for $V$ controls for the component of $\theta_1$ in $(U-1)'$ that gives rise to the spurious dependence. The Blundell and Powell (2003, 2004) application of the control function principle assumes functional form (6) but assumes that $V$ can be perfectly proxied by a first stage equation. Thus they use a replacement function in their first stage. Their method does not work when one can only condition on $D$ rather than on $D^* = \mu_D(Z) - V$ instead of directly measuring it.\footnote{Imbens and Newey (2002) extend their approach. See the discussion in Matzkin (2007).}

In the sample selection model, it is not necessary to identify $V$. As developed in Heckman and Robb (1985a) and Heckman and Vytlacil (2007a,b), under additive separability for the outcome equation for $Y_1$, one can write

$$E(Y_1 \mid X, Z, D = 1) = \mu_1(X) + E(U_1 \mid \mu_D(Z) > V),$$

so the analyst “expects out” rather than solve out the effect of the component of $V$ on $U_1$ and thus control for selection bias under the maintained assumptions. In terms of the propensity score, under the conditions specified in Heckman and Vytlacil (2007a), one may write the preceding expression in terms of $P(Z)$:

$$E(Y_1 \mid X, Z, D = 1) = \mu_1(X) + K_1(P(Z)),$$

where $K_1(P(Z)) = E(U_1 \mid X, Z, D = 1)$. It is not literally necessary to know $V$ or be able to estimate it. The Blundell and Powell (2003, 2004) application of the control function principle assumes that the analyst can condition on and estimate $V$.

The Blundell-Powell method and the method of Imbens and Newey (2002) build heavily on (6) and implicitly make strong distributional and functional form assumptions that are not intrinsic to the method of control functions. As just noted, their method uses a replacement function to obtain $E(U_1 \mid V)$ in the first step of their procedures. The general control function method does not require a replacement function approach. The literature has begun to distinguish between the more general control function approach and the control variate approach that uses a first stage replacement function.
Matzkin (2003) develops the method of unobservable instruments which is a version of the replacement function approach applied to nonlinear models. Her unobservable instruments play the role of covariance restrictions used to identify classical simultaneous equations models (see Fisher, 1966). Her approach is distinct from and therefore complementary with linear factor models. Instead of assuming \((X, Z, Q) \perp \perp (\theta, \epsilon_j)\), she assumes in a two equation system that \((\theta, \epsilon_1) \perp \perp Y_2 \mid Y_1, X, Z\). See Matzkin (2007).

I do not discuss panel data methods in this paper. The most commonly used panel data method is difference-in-differences as discussed in Heckman and Robb (1985a), Blundell, Duncan, and Meghir (1998), Heckman, LaLonde, and Smith (1999), and Bertrand, Duflo, and Mullainathan (2004), to cite only a few of the key papers. Most of the estimators I have discussed can be adapted to a panel data setting. Heckman, Ichimura, Smith, and Todd (1998) develop difference-in-differences matching estimators. Abadie (2002) extends this work.\textsuperscript{23} Separability between errors and observables is a key feature of the panel data approach in its standard application. Altonji and Matzkin (2005) and Matzkin (2003) present analyses of nonseparable panel data methods. Regression discontinuity estimators, which are versions of IV estimators, are discussed by Heckman and Vytlacil (2007b).

Table 1 summarizes some of the main lessons of this section. I stress that the stated conditions are necessary conditions. There are many versions of the IV and control functions principle and extensions of these ideas which refine these basic postulates. See Heckman and Vytlacil (2007b). Matzkin (2007) is an additional reference on sources of identification in econometric models.

I next turn to a detailed discussion of the method of matching which currently is “the method of choice” for evaluating social programs in Europe. When its strong assumptions and severe limitations are understood, perhaps its popularity will wane. Before analyzing matching, it is useful to have a prototypical choice model in hand. I introduce the generalized Roy model and the concept of the marginal treatment effect which helps to understand the strong assumptions made in the method of matching.

\textsuperscript{23}There is related work by Athey and Imbens (2006).
2 A Prototypical Policy Evaluation Problem

To motivate my discussion in the rest of this paper, consider the following prototypical policy problem. Suppose a policy is proposed for adoption in a country. It has been tried in other countries and we know outcomes there. We also know outcomes in countries where it was not adopted. From the historical record, what can we conclude about the likely effectiveness of the policy in countries that have not implemented it?

To answer questions of this sort, economists build models of counterfactuals. Consider the following model. Let $Y_0$ be the outcome of a country (e.g. GDP) under a no-policy regime. $Y_1$ is the outcome if the policy is implemented. $Y_1 - Y_0$ is the “treatment effect” of the policy. It may vary among countries. We observe characteristics $X$ of various countries (e.g. level of democracy, level of population literacy, etc.). It is convenient to decompose $Y_1$ into its mean given $X$, $\mu_1(X)$ and deviation from mean $U_1$. One can make a similar decomposition for $Y_0$:

$$Y_1 = \mu_1(X) + U_1$$
$$Y_0 = \mu_0(X) + U_0.$$  \hspace{1cm} (7)

Additive separability is not needed, but it is convenient to assume it, and I initially adopt it to simplify the exposition and establish a parallel regression notation that serves to link the statistical literature on treatment effects with the economic literature.

It may happen that controlling for the $X$, $Y_1 - Y_0$ is the same for all countries. This is the case of homogeneous treatment effects given $X$. More likely, countries vary in their responses to the policy even after controlling for $X$.

Figure 1 plots the distribution of $Y_1 - Y_0$ for a benchmark $X$. It also displays the various conventional treatment parameters. I use a special form of the generalized Roy model with constant cost $C$ of adopting the policy. This is called the “extended Roy model”. I use this model because it is simple and intuitive. (The precise parameterization of the extended Roy model used to generate the figure and the treatment effects is given at the base of figure 1.) The special case of homogeneity in $Y_1 - Y_0$ arises when the distribution collapses to its mean. It would be ideal if
one could estimate the distribution of $Y_1 - Y_0$ given $X$ and there is research that does this.

More often, economists focus on some mean of the distribution displayed in figure 1 and use a regression framework to interpret the data. To turn (7) into a regression model, it is conventional to use the switching regression framework. Define $D = 1$ if a country adopts a policy; $D = 0$ if it does not. The observed outcome $Y$ is the switching regression model (1). Substituting (7) into this expression, and keeping all $X$ implicit, one obtains

$$Y = Y_0 + (Y_1 - Y_0)D \quad \text{(8)}$$

$$= \mu_0 + (\mu_1 - \mu_0 + U_1 - U_0)D + U_0.$$ 

Using conventional regression notation,

$$Y = \alpha + \beta D + \varepsilon \quad \text{(9)}$$

where $\alpha = \mu_0$, $\beta = (Y_1 - Y_0) = \mu_1 - \mu_0 + U_1 - U_0$ and $\varepsilon = U_0$. I will also use the notation that $v = U_1 - U_0$, letting $\bar{\beta} = \mu_1 - \mu_0$ and $\beta = \bar{\beta} + v$. Throughout this paper I use treatment effect and regression notation interchangeably. The coefficient on $D$ is the treatment effect. The case where $\beta$ is the same for every country is the case conventionally assumed. More elaborate versions assume that $\beta$ depends on $X$ ($\beta(X)$) and estimates interactions of $D$ with $X$. The case where $\beta$ varies even after accounting for $X$ is called the “random coefficient” or “heterogenous treatment effect” case. The case where $v = U_1 - U_0$ depends on $D$ is the case of essential heterogeneity analyzed by Heckman, Urzua, and Vytlacil (2006). This case arises when treatment choices depend at least in part on the idiosyncratic return to treatment. A great deal of attention has been focused on this case in recent decades and I develop the implications of this model in this paper.

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\footnote{Statisticians sometimes attribute this representation to Rubin (1974, 1978), but it is due to Quandt (1958, 1972). It is implicit in the Roy (1951) model. See the discussion of this basic model of counterfactuals in Heckman and Vytlacil (2007a).}
2.1 An Index Model of Choice and Treatment Effects: Definitions and Unifying Principles

I now present the model of treatment effects developed in Heckman and Vytlacil (1999, 2001, 2005, 2007a,b) and Heckman, Urzua, and Vytlacil (2006), which relaxes the normality, separability and exogeneity assumptions invoked in the traditional economic selection models. It is rich enough to generate all of the treatment effects displayed in figure 2 as well as many other policy parameters. It does not require separability. It is a nonparametric generalized Roy model with testable restrictions that can be used to unify the treatment effect literature, identify different treatment effects, link the literature on treatment effects to the literature in structural econometrics and interpret the implicit economic assumptions underlying instrumental variables, regression discontinuity design methods, control functions and matching methods.

$Y$ is the measured outcome variable. It is produced from the switching regression model (1). Outcomes are general nonlinear, nonseparable functions of observables and unobservables:

\[
Y_1 = \mu_1(X, U_1) \quad \text{(10)}
\]

\[
Y_0 = \mu_0(X, U_0). \quad \text{(11)}
\]

Examples of models that can be written in this form include conventional latent variable models for discrete choice that are generated by a latent variable crossing a threshold: $Y_i = 1(Y^*_i \geq 0)$, where $Y^*_i = \mu_i(X) + U_i, i = 0, 1$. Notice that in the general case, $\mu_i(X, U_i) - E(Y_i \mid X) \neq U_i, i = 0, 1$.

The individual treatment effect associated with moving an otherwise identical person from “0” to “1” is $Y_1 - Y_0 = \Delta$ and is defined as the causal effect on $Y$ of a ceteris paribus move from “0” to “1”. To link this framework to the literature on economic choice models, I characterize the decision rule for program participation by an index model:

\[
D^* = \mu_D(Z) - V; \quad D = 1 \text{ if } D^* \geq 0; \quad D = 0 \text{ otherwise}, \quad \text{(12)}
\]

where, from the point of view of the econometrician, $(Z, X)$ is observed and $(U_1, U_0, V)$ is unobserved. The random variable $V$ may be a function of $(U_1, U_0)$. For example, in the original Roy
Model, $\mu_1$ and $\mu_0$ are additively separable in $U_1$ and $U_0$ respectively, and $V = -[U_1 - U_0]$. In the original formulations of the generalized Roy model, outcome equations are separable and $V = -[U_1 - U_0 - U_C]$, where $U_C$ arises from the cost function. Without loss of generality, I define $Z$ so that it includes all of the elements of $X$ as well as any additional variables unique to the choice equation.

I invoke the following assumptions that are weaker than those used in the conventional literature on structural econometrics or the recent literature on semiparametric selection models and at the same time can be used both to define and to identify different treatment parameters.\(^{25}\) The assumptions are:

(A-1) $(U_0, U_1, V)$ are independent of $Z$ conditional on $X$ (Independence);

(A-2) $\mu_D(Z)$ is a nondegenerate random variable conditional on $X$ (Rank Condition);

(A-3) The distribution of $V$ is continuous;\(^{26}\)

(A-4) The values of $E|Y_1|$ and $E|Y_0|$ are finite (Finite Means);

(A-5) $0 < \Pr(D = 1 | X) < 1$.

(A-1) assumes that $V$ is independent of $Z$ given $X$ and is used below to generate counterfactuals. For the definition of treatment effects one does not need either (A-1) or (A-2). The definitions of treatment effects and their unification do not require any elements of $Z$ that are not elements of $X$ or independence assumptions. However, an analysis of instrumental variables requires that $Z$ contain at least one element not in $X$. Assumptions (A-1) or (A-2) justify application of instrumental variables methods and nonparametric selection or control function methods. Some parameters in the recent IV literature are defined by an instrument so I make assumptions about instruments up front, noting where they are not needed. Assumption (A-4) is needed to satisfy standard integration conditions. It guarantees that the mean treatment parameters are well defined. Assumption (A-5) is the assumption in the population of both a treatment and a control group for each $X$. Observe that there are no exogeneity requirements for $X$. This is in

\(^{25}\) A much weaker set of conditions is required to define the parameters than is required to identify them. See Heckman and Vytlacil (2007b, appendix B).

\(^{26}\) Absolutely continuous with respect to Lebesgue measure.
contrast with the assumptions commonly made in the conventional structural literature and the semiparametric selection literature (see, e.g. Powell, 1994).

A counterfactual “no feedback” condition facilitates interpretability so that conditioning on $X$ does not mask the effects of $D$. Letting $X_d$ denote a value of $X$ if $D$ is set to $d$, a sufficient condition that rules out feedback from $D$ to $X$ is:

\[(A-6) \text{ Let } X_0 \text{ denote the counterfactual value of } X \text{ that would be observed if } D \text{ is set to } 0. \text{ } X_1 \text{ is defined analogously. Assume } X_d = X \text{ for } d = 0, 1. (The } X_D \text{ are invariant to counterfactual manipulations.)}\]

Condition (A-6) is not strictly required to formulate an evaluation model, but it enables an analyst who conditions on $X$ to capture the “total” or “full effect” of $D$ on $Y$ (see Pearl, 2000). This assumption imposes the requirement that $X$ is an external variable determined outside the model and is not affected by counterfactual manipulations of $D$. However, the assumption allows for $X$ to be freely correlated with $U_1, U_0$ and $V$ so it can be endogenous.

In this notation, $P(Z)$ is the probability of receiving treatment given $Z$, or the “propensity score” $P(Z) = \Pr(D = 1 \mid Z) = F_{V \mid X}(\mu_D(Z))$, where $F_{V \mid X}(\cdot)$ denotes the distribution of $V$ conditional on $X$.\(^{27}\) I denote $P(Z)$ by $P$, suppressing the $Z$ argument. I also work with $U_D$, a uniform random variable ($U_D \sim \text{Unif}[0, 1]$) defined by $U_D = F_{V \mid X}(V)$.\(^{28}\) The separability between $V$ and $\mu_D(Z)$ or $D(Z)$ and $U_D$ is conventional. It plays a crucial role in justifying instrumental variable estimators in the general models analyzed in this paper.

Vytlacil (2002) establishes that assumptions (A-1)–(A-5) for selection model (1) and (10)–(12) are equivalent to the assumptions used to generate the LATE model of Imbens and Angrist (1994). Thus the nonparametric selection model for treatment effects developed by Heckman and Vytlacil is implied by the assumptions of the Imbens-Angrist instrumental variable model for treatment effects. Their approach is more general and links the IV literature to the literature on economic choice models. The latent variable model is a version of the standard sample selection

\(^{27}\)Throughout this paper, I will refer to the cumulative distribution function of a random vector $A$ by $F_A(\cdot)$ and to the cumulative distribution function of a random vector $A$ conditional on random vector $B$ by $F_{A \mid B}(\cdot)$. I will write the cumulative distribution function of $A$ conditional on $B = b$ by $F_{A \mid B}(\cdot \mid b)$.

\(^{28}\)This representation is valid whether or not (A-1) is true. However, (A-1) imposes restrictions on counterfactual choices. For example, if a change in government policy changes the distribution of $Z$ by an external manipulation, under (A-1) the model can be used to generate the choice probability from $P(z)$ evaluated at the new arguments, i.e., the model is invariant with respect to the distribution $Z$. 

23
bias model. This weaves together two strands of the literature often thought to be distinct (see e.g. Angrist and Krueger, 1999). Heckman, Urzua, and Vytlacil (2006) develop this parallelism in detail.\textsuperscript{29}

\subsection*{2.2 Definitions of Treatment Effects in the Two Outcome Model}

The difficulty of observing the same individual in both treated and untreated states leads to the use of various population level treatment effects widely used in the biostatistics literature and often applied in economics.\textsuperscript{30} The most commonly invoked treatment effect is the Average Treatment Effect (ATE): \( \Delta_{ATE}(x) \equiv E(\Delta \mid X = x) \) where \( \Delta = Y_1 - Y_0 \). This is the effect of assigning treatment randomly to everyone of type \( X \) assuming full compliance, and ignoring general equilibrium effects.\textsuperscript{31} The average impact of treatment on persons who actually take the treatment is Treatment on the Treated (TT): \( \Delta_{TT}(x) \equiv E(\Delta \mid X = x, D = 1) \). This parameter can also be defined conditional on \( P(Z) \): \( \Delta_{TT}(x, p) \equiv E(\Delta \mid X = x, P(Z) = p, D = 1) \).\textsuperscript{32}

The mean effect of treatment on those for whom \( X = x \) and \( U_D = u_D \), the Marginal Treatment Effect (MTE), plays a fundamental role in the analysis of the next section:

\[ \Delta_{MTE}(x, u_D) \equiv E(\Delta \mid X = x, U_D = u_D). \] (13)

This parameter is defined independently of any instrument. I separate the definition of parameters from their identification. The MTE is the expected effect of treatment conditional on observed

\textsuperscript{29}The model of equations (10)-(12) and assumptions (A-1)–(A-5) impose two testable restrictions on the distribution of \((Y, D, Z, X)\). First, it imposes an index sufficiency restriction: for any set \( \mathcal{A} \) and for \( j = 0, 1 \),

\[ \Pr(Y_j \in \mathcal{A} \mid X, Z, D = j) = \Pr(Y_j \in \mathcal{A} \mid X, P(Z), D = j). \]

\( Z \) (given \( X \)) enters the model only through the propensity score \( P(Z) \) (the sets of \( \mathcal{A} \) are assumed to be measurable). This restriction has empirical content when \( Z \) contains two or more variables not in \( X \). Second, the model also imposes monotonicity in \( p \) for \( E(YD \mid X = x, P = p) \) and \( E(Y(1-D) \mid X = x, P = p) \). Heckman and Vytlacil (2005, appendix A) develop this condition further, and show that it is testable.

Even though this model of treatment effects is not the most general possible model, it has testable implications and hence empirical content. It unites various literatures and produces a nonparametric version of the selection model, and links the treatment literature to economic choice theory.

\textsuperscript{30}Heckman, LaLonde, and Smith (1999) discuss panel data cases where it is possible to observe both \( Y_0 \) and \( Y_1 \) for the same person.

\textsuperscript{31}See, e.g., Imbens (2004).

\textsuperscript{32}These two definitions of treatment on the treated are related by integrating out the conditioning \( p \) variable: \( \Delta_{TT}(x) = \int_0^1 \Delta_{TT}(x, p) dF_{P(Z)}(p|x, 1) \) where \( F_{P(Z)}(\cdot|x, 1) \) is the distribution of \( P(Z) \) given \( X = x \) and \( D = 1 \).
characteristics $X$ and conditional on $U_D$, the unobservables from the first stage decision rule. For $u_D$ evaluation points close to zero, $\Delta^{\text{MTE}}(x, u_D)$ is the expected effect of treatment on individuals with the value of unobservables that make them most likely to participate in treatment and who would participate even if the mean scale utility $\mu_D(Z)$ is small. If $U_D$ is large, $\mu_D(Z)$ would have to be large to induce people to participate.

One can also interpret $E(\Delta | X = x, U_D = u_D)$ as the mean gain in terms of $Y_1 - Y_0$ for persons with observed characteristics $X$ who would be indifferent between treatment or not if they were randomly assigned a value of $Z$, say $z$, such that $\mu_D(z) = u_D$. When $Y_1$ and $Y_0$ are value outcomes, MTE is a mean willingness-to-pay measure. MTE is a choice-theoretic building block that unites the treatment effect, selection, matching and control function literatures.

A third interpretation is that MTE conditions on $X$ and the residual defined by subtracting the expectation of $D^*$ from $D^*$: $\bar{U}_D = D^* - E(D^* | Z, X)$. This is a “replacement function” interpretation in the sense of Heckman and Robb (1985a) and Matzkin (2007), or “control function” interpretation in the sense of Blundell and Powell (2003). The additive separability of equation (12) in terms of observables and unobservables plays a crucial role in the justification of instrumental variable methods.

The LATE parameter of Imbens and Angrist (1994) is a version of MTE. I define LATE independently of any instrument after first presenting the Imbens–Angrist definition. Define $D(z)$ as a counterfactual choice variable, with $D(z) = 1$ if $D$ would have been chosen if $Z$ had been set to $z$, and $D(z) = 0$ otherwise. Let $\mathcal{Z}(x)$ denote the support of the distribution of $Z$ conditional on $X = x$. For any $(z, z') \in \mathcal{Z}(x) \times \mathcal{Z}(x)$ such that $P(z) > P(z')$, LATE is $E(\Delta | X = x, D(z) = 1, D(z') = 0) = E(Y_1 - Y_0 | X = x, D(z) = 1, D(z') = 0)$, the mean gain to persons who would be induced to switch from $D = 0$ to $D = 1$ if $Z$ were manipulated externally from $z'$ to $z$. In an example of the returns to education, $z'$ could be the base level of tuition and $z$ a reduced tuition level. Using the latent index model, Heckman and Vytlacil (1999, 2005) show that LATE

---

These three interpretations are equivalent under separability in $D^*$, i.e., when (12) characterizes the choice equation, but lead to three different definitions of MTE when a more general nonseparable model is developed. See Heckman and Vytlacil (2007b).
can be written as

\[ E(Y_1 - Y_0 \mid X = x, D(z) = 1, D(z') = 0) = E(Y_1 - Y_0 \mid X = x, u_D' < U_D < u_D) = \Delta^{\text{LATE}}(x, u_D, u_D') \]

for \( u_D = \Pr(D(z) = 1) = P(z), \ u_D' = \Pr(D(z') = 1) = P(z') \), where assumption (A-1) implies that \( \Pr(D(z) = 1) = \Pr(D = 1 \mid Z = z) \) and \( \Pr(D(z') = 1) = \Pr(D = 1 \mid Z = z') \).

Imbens and Angrist define the LATE parameter as the probability limit of an estimator. Their analysis conflates issues of definition of parameters with issues of identification. The representation of LATE given here allows analysts to separate these two conceptually distinct matters and to define the LATE parameter more generally. One can in principle evaluate the right hand side of the preceding equation at any \( u_D, u_D' \) points in the unit interval and not only at points in the support of the distribution of the propensity score \( P(Z) \) conditional on \( X = x \) where it is identified. From assumptions (A-1), (A-3), and (A-4), \( \Delta^{\text{LATE}}(x, u_D, u_D') \) is continuous in \( u_D \) and \( u_D' \) and

\[ \lim_{u_D' \uparrow u_D} \Delta^{\text{LATE}}(x, u_D, u_D') = \Delta^{\text{MTE}}(x, u_D). \]

Heckman and Vytlacil (1999) use assumptions (A-1)–(A-5) and the latent index structure to develop the relationship between MTE and the various treatment effect parameters shown in the first three lines of table 2A. They present the formal derivation of the parameters and associated weights and graphically illustrates the relationship between ATE and TT. All treatment parameters may be expressed as weighted averages of the MTE:

\[ \text{Treatment Parameter } (j) = \int \Delta^{\text{MTE}}(x, u_D) \omega_j(x, u_D) \, du_D \]

where \( \omega_j(x, u_D) \) is the weighting function for the MTE and the integral is defined over the full support of \( u_D \). Except for the OLS weights, the weights in the table all integrate to one, although in some cases the weights for IV may be negative (Heckman, Urzua, and Vytlacil, 2006).

\[ ^{34} \text{This follows from Lebesgue’s theorem for the derivative of an integral and holds almost everywhere with respect to Lebesgue measure. The ideas of the marginal treatment effect and the limit form of LATE were first introduced in the context of a parametric normal generalized Roy model by Björklund and Moffitt (1987), and were analyzed more generally in Heckman (1997). Angrist, Graddy, and Imbens (2000) also define and develop a limit form of LATE.} \]
In table 2A, $\Delta^{TT}(x)$ is shown as a weighted average of $\Delta^{MTE}$:

$$\Delta^{TT}(x) = \int_0^1 \Delta^{MTE}(x, u_D) \omega_{TT}(x, u_D) \, du_D,$$

where

$$\omega_{TT}(x, u_D) = \frac{1 - F_{P|X}(u_D | x)}{\int_0^1 (1 - F_{P|X}(t | x)) \, dt} = \frac{S_{P|X}(u_D | x)}{E(P(Z) | X = x)},$$

and $S_{P|X}(u_D | x)$ is $\Pr(P(Z) > u_D | X = x)$ and $\omega_{TT}(x, u_D)$ is a weighted distribution. The parameter $\Delta^{TT}(x)$ oversamples $\Delta^{MTE}(x, u_D)$ for those individuals with low values of $u_D$ that make them more likely to participate in the program being evaluated. Treatment on the untreated (TUT) is defined symmetrically with TT and oversamples those least likely to participate. The various weights are displayed in table 2B. A central theme of the analysis of Heckman and Vytlacil is that under their assumptions all estimators and estimands can be written as weighted averages of MTE. This allows them to unify the treatment effect literature using a common functional MTE ($u_D$).

Observe that if $E(Y_1 - Y_0 | X = x, U_D = u_D) = E(Y_1 - Y_0 | X = x)$, so $\Delta = Y_1 - Y_0$ is mean independent of $U_D$ given $X = x$, then $\Delta^{MTE} = \Delta^{ATE} = \Delta^{TT} = \Delta^{LATE}$. Therefore in cases where there is no heterogeneity in terms of unobservables in MTE ($\Delta$ constant conditional on $X = x$) or agents do not act on it so that $U_D$ drops out of the conditioning set, marginal treatment effects are average treatment effects, so that all of the evaluation parameters are the same. Otherwise, they are different. Only in the case where the marginal treatment effect is the average treatment effect will the “effect” of treatment be uniquely defined.

Figure 2A plots weights for a parametric normal generalized Roy model generated from the parameters shown at the base of figure 2B. The model allows for costs to vary in the population and is more general than the extended Roy model used to construct figure 1. The weights for IV depicted in figure 2B are discussed in Heckman, Urzua, and Vytlacil (2006) and the weights for OLS are discussed in the next section. A high $u_D$ is associated with higher cost, relative to return, and less likelihood of choosing $D = 1$. The decline of MTE in terms of higher values of $u_D$ means that people with higher $u_D$ have lower gross returns. TT overweights low values of $u_D$ (i.e., it oversamples $U_D$ that make it likely to have $D = 1$). ATE samples $U_D$ uniformly. Treatment on
the Untreated \((E(Y_1 - Y_0 \mid X = x, D = 0))\), or TUT, oversamples the values of \(U_D\) which make it unlikely to have \(D = 1\).

Table 3 shows the treatment parameters produced from the different weighting schemes for the model used to generate the weights in figures 2A and 2B. Given the decline of the MTE in \(u_D\), it is not surprising that TT>ATE>TUT. This is the generalized Roy version of the principle of diminishing returns. Those most likely to self select into the program benefit the most from it. The difference between TT and ATE is a sorting gain: \(E(Y_1 - Y_0 \mid X, D = 1) - E(Y_1 - Y_0 \mid X)\), the average gain experienced by people who sort into treatment compared to what the average person would experience. Purposive selection on the basis of gains should lead to positive sorting gains of the kind found in the table. If there is negative sorting on the gains, then TUT ≥ ATE ≥ TT.

2.3 The Weights for a Generalized Roy Model

For the case of continuous \(Z\), I plot the weights associated with the MTE for IV. This analysis draws on Heckman, Urzua, and Vytlacil (2006), who derive the weights. Figure 3 plots \(E(Y \mid P(Z))\) and MTE for the extended Roy models generated by the parameters displayed at the base of the figure. In cases where \(\beta \perp D\), \(\Delta^\text{MTE}(u_D)\) is constant in \(u_D\). This is trivial when \(\beta\) is a constant. When \(\beta\) is random but selection into \(D\) does not depend on \(\beta\), MTE is still flat. The more interesting case termed “essential heterogeneity” by Heckman and Vytlacil has \(\beta \not\perp D\). The left hand side (figure 3A) depicts \(E(Y \mid P(Z))\) in the two cases. The first case makes \(E(Y \mid P(Z))\) linear in \(P(Z)\). The second case is nonlinear in \(P(Z)\). This arises when \(\beta \not\perp D\). The derivative of \(E(Y \mid P(Z))\) is presented in the right panel (figure 3B). It is a constant for the first case (flat MTE) but declining in \(U_D = P(Z)\) for the case with selection on the gain. A simple test for linearity in \(P(Z)\) in the outcome equation reveals whether or not the analyst is in cases I and II (\(\beta \perp D\)) or case III (\(\beta \not\perp D\)). These cases are the extended Roy counterparts to \(E(Y \mid P(Z) = p)\) and MTE shown for the generalized Roy model in figures 4A and 4B.

MTE gives the mean marginal return for persons who have utility \(P(Z) = u_D\). Thus, \(P(Z) = u_D\) is the margin of indifference. Those with low \(u_D\) values have high returns. Those with high \(u_D\)

\[^{35}\text{Recall that we keep the conditioning on } X \text{ implicit.}\]
values have low returns. Figure 3 highlights that in the general case MTE (and LATE) identify average returns for persons at the margin of indifference at different levels of the mean utility function \((P(Z))\).

Figure 5 plots MTE and LATE for different intervals of \(u_D\) using the model generating figure 3. LATE is the chord of \(E(Y \mid P(Z))\) evaluated at different points. The relationship between LATE and MTE is depicted in the right panel (B) of figure 5. LATE is the integral under the MTE curve divided by the difference between the upper and lower limits.

Treatment parameters associated with the second case are plotted in figure 6. The MTE is the same as that presented in figure 3. ATE has the same value for all \(p\). The effect of treatment on the treated for \(P(Z) = p, \Delta TT(p) = E(Y_1 - Y_0 \mid D = 1, P(Z) = p)\) declines in \(p\) (equivalently it declines in \(u_D\)). Treatment on the untreated given \(p\), \(\text{TUT}(p) = \Delta TUT(p) = E(Y_1 - Y_0 \mid D = 0, P(Z) = p)\) also declines in \(p\).

\[
\begin{align*}
\text{LATE}(p, p') &= \frac{\Delta TT(p')p' - \Delta TT(p)p}{p' - p}, \quad p' \neq p \\
MTE &= \frac{\partial[\Delta TT(p)p]}{\partial p}.
\end{align*}
\]

One can generate all of the treatment parameters from \(\Delta TT(p)\).

Matching on \(P = p\) (which is equivalent to nonparametric regression given \(P = p\) produces a biased estimator of \(TT(p)\). Matching assumes a flat MTE (average return equals marginal return) as we develop below. Therefore it is systematically biased for \(\Delta TT(p)\) in a model with essential heterogeneity, where \(\beta \perp D\). Making observables alike makes the unobservables dissimilar. Holding \(p\) constant across treatment and control groups understates \(TT(p)\) for low values of \(p\) and overstates it for high values of \(p\). I develop this point further in the next section, where I discuss the method of matching.
3 Matching

The method of matching assumes selection of treatment based on potential outcomes

\[(Y_0, Y_1) \not \perp D,\]

so \(\Pr(D = 1 \mid Y_0, Y_1)\) depends on \(Y_0, Y_1\). It assumes access to variables \(Q\) such that conditioning on \(Q\) removes the dependence:

\[(Y_0, Y_1) \perp D \mid Q.\]  \hspace{1cm} (Q-1)

Thus,

\[
\Pr(D = 1 \mid Q, Y_0, Y_1) = \Pr(D = 1 \mid Q).
\]

Comparisons between treated and untreated can be made at all points in the support of \(Q\) such that

\[0 < \Pr(D = 1 \mid Q) < 1.\] \hspace{1cm} (Q-2)

The method does not explicitly model choices of treatment or the subjective evaluations of participants, nor is there any distinction between the variables in the outcome equations \((X)\) and the variables in the choice equations \((Z)\) that is central to the IV method and the method of control functions. In principle, condition (Q-1) can be satisfied using a set of variables \(Q\) distinct from all or some of the components of \(X\) and \(Z\). The conditioning variables do not have to be exogenous.

From condition (Q-1) one recovers the distributions of \(Y_0\) and \(Y_1\) given \(Q\)—\(\Pr(Y_0 \leq y_0 \mid Q = q) = F_0(y_0 \mid Q = q)\) and \(\Pr(Y_1 \leq y_1 \mid Q = q) = F_1(y_1 \mid Q = q)\)—but not the joint distribution \(F_{0,1}(y_0, y_1 \mid Q = q)\), because the analyst does not observe the same persons in the treated and untreated states. This is a standard evaluation problem common to all econometric estimators. Methods for determining which variables belong in \(Q\) rely on untested exogeneity assumptions which we discuss in this section.

OLS is a special case of matching that focuses on the identification of conditional means. In OLS linear functional forms are maintained as exact representations or valid approximations.
Considering a common coefficient model, OLS writes

\[ Y = \pi Q + D\alpha + U, \quad (Q-3) \]

where \( \alpha \) is the treatment effect and

\[ E(U \mid Q, D) = 0. \quad (Q-4) \]

The assumption is made that the variance-covariance matrix of \((Q, D)\) is of full rank:

\[ \text{Var}(Q, D) \text{ full rank.} \quad (Q-5) \]

Under these conditions, one can identify \( \alpha \) even though \( D \) and \( U \) are dependent: \( D \not\perp U \). Controlling for the observable \( Q \) eliminates any spurious mean dependence between \( D \) and \( U \): \( E(U \mid D) \neq 0 \) but \( E(U \mid D, Q) = 0 \). \( (Q-3) \) is the linear regression counterpart to \((Q-1)\). \( (Q-5) \) is the linear regression counterpart to \((Q-2)\). Failure of \( (Q-5) \) would mean that using a nonparametric estimator one might perfectly predict \( D \) given \( Q \), and that \( \Pr(D = 1 \mid Q = q) = 1 \) or \( 0 \).\(^{36}\)

Matching can be implemented as a nonparametric method. When this is done, the procedure does not require specification of the functional form of the outcome equations. It enforces the requirement that \((Q-2)\) be satisfied by estimating functions pointwise in the support of \( Q \). Assume that \( Q = (X, Z) \) and that \( X \) and \( Z \) are the same except where otherwise noted. Thus I invoke assumptions \((M-1)\) and \((M-2)\) presented in section 1, even though in principle one can use a more general conditioning set.

Assumptions \((M-1)\) and \((M-2)\) or \((Q-1)\) and \((Q-2)\) rule out the possibility that after conditioning on \( X \) (or \( Q \)), agents possess more information about their choices than econometricians, and that the unobserved information helps to predict the potential outcomes. Put another way, the method allows for potential outcomes to affect choices but only through the observed variables, \( Q \), that predict outcomes. This is the reason why Heckman and Robb (1985a, 1986b) call the method selection on observables.

\(^{36}\)This condition might be met only at certain values of \( Q = q \). For certain parameterizations (e.g. the linear probability model), one may obtain predicted probabilities outside the unit interval.
This section establishes the following points. (1) Matching assumptions (M-1) and (M-2) generically imply a flat MTE in $u_D$, i.e. they assume that \( E(Y_1 - Y_0 \mid X = x, U_D = u_D) \) does not depend on $u_D$. Thus the unobservables central to the Roy model and its extensions and the unobservables central to the modern IV literature are assumed to be absent once the analyst conditions on $X$. (M-1) implies that all mean treatment parameters are the same. (2) Even if one weakens (M-1) and (M-2) to mean independence instead of full independence, generically the MTE is flat in $u_D$ under the assumptions of the nonparametric generalized Roy model developed in section 2.1, so again all mean treatment parameters are the same. (3) I show that IV and matching make distinct identifying assumptions even though they both invoke conditional independence assumptions. (4) I compare matching with IV and control function (sample selection) methods. Matching assumes that conditioning on observables eliminates the dependence between $(Y_0, Y_1)$ and $D$. The control function principle models the dependence. (5) I present some examples that demonstrate that if the assumptions of the method of matching are violated, the method can produce substantially biased estimators of the parameters of interest. (6) I show that standard methods for selecting the conditioning variables used in matching assume exogeneity. Violations of the exogeneity assumption can produce biased estimators.

Nonparametric versions of matching embodying (M-2) avoid the problem of making inferences outside the support of the data. This problem is implicit in any application of least squares. Figure 7 shows the support problem that can arise in linear least squares when the linearity of the regression is used to extrapolate estimates determined in one empirical support to new supports. Careful attention to support problems is a virtue of any nonparametric method, including, but not unique to, nonparametric matching. Heckman, Ichimura, Smith, and Todd (1998) show that the bias from neglecting the problem of limited support can be substantial. See also the discussion in Heckman, LaLonde, and Smith (1999).

I now show that matching implies that conditional on $X$, the marginal return is assumed to be the same as the average return (marginal $=$ average). This is a strong behavioral assumption implicit in statistical conditional independence assumption (M-1). It says that the marginal participant has the same return as the average participant.
3.1 Matching Assumption (M-1) Implies a Flat MTE

An immediate consequence of (M-1) is that the MTE does not depend on $U_D$. This is so because $(Y_0, Y_1) \perp D | X$ implies that $(Y_0, Y_1) \perp U_D | X$ and hence that

$$\Delta_{\text{MTE}}(x, u_D) = E(Y_1 - Y_0 | X = x, U_D = u_D) = E(Y_1 - Y_0 | X = x).$$  

(15)

This, in turn, implies that $\Delta_{\text{MTE}}$ conditional on $X$ is flat in $u_D$. Under the stated assumptions for the generalized Roy model, it assumes that $E(Y | P(Z) = p)$ is linear in $p$. Thus the method of matching assumes that mean marginal returns and average returns are the same and all mean treatment effects are the same given $X$. However, one can still distinguish marginal from average effects of the observables ($X$) using matching. See Carneiro (2002).

It is sometimes said that the matching assumptions are “for free” (See, e.g., Gill and Robins, 2001) because one can always replace unobserved $F_1(Y_1 | X = x, D = 0)$ with observed $F_1(Y_1 | X = x, D = 1)$ and unobserved $F_0(Y_0 | X = x, D = 1)$ with observed $F_0(Y_0 | X = x, D = 0)$. Such replacements do not contradict any observed data.

While the claim is true, it ignores the counterfactual states generated under the matching assumptions. The assumed absence of selection on unobservables is not a “for free” assumption, and produces fundamentally different counterfactual states for the same model under matching and selection assumptions. To explore these issues in depth, consider a nonparametric regression model more general than the linear regression model (Q-3).

Without assumption (M-1), a nonparametric regression of $Y$ on $D$ conditional on $X$ identifies a nonparametric mean difference:

$$\Delta_{\text{OLS}}(X) = E(Y_1 | X, D = 1) - E(Y_0 | X, D = 0)$$

$$= E(Y_1 - Y_0 | X, D = 1) + \{E(Y_0 | X, D = 1) - E(Y_0 | X, D = 0)\}. \quad (16)$$

The term in braces in the second expression arises from selection on pre-treatment levels of the outcome. OLS identifies the parameter treatment on the treated (the first term in the second line of (16)) plus a bias term in braces corresponding to selection on the levels.
The OLS estimator can be represented as a weighted average of $\Delta^{MTE}$. The weight is given in table 2B where $U_1$ and $U_0$ for the OLS model are defined as deviations from conditional expectations, $U_1 = Y_1 - E(Y_1 \mid X)$, $U_0 = Y_0 - E(Y_0 \mid X)$. Unlike the weights for $\Delta^{TT}$ and $\Delta^{ATE}$, the OLS weights do not necessarily integrate to one and they are not necessarily nonnegative. Under its assumptions, application of IV eliminates the contribution of the second term of equation (16). The weights for the first term are the same as the weights for $\Delta^{TT}$ and hence they integrate to one.

The OLS weights for the generalized Roy model example are plotted in figure 2B. The negative component of the OLS weight leads to a smaller OLS treatment estimate compared to the other treatment effects in table 3. This table shows the estimated OLS treatment effect for a generalized Roy example. The large negative selection bias in this example is consistent with comparative advantage as emphasized by Roy (1951) and detected empirically by Willis and Rosen (1979) and Cunha, Heckman, and Navarro (2005). People who are good in sector 1 (i.e. receive treatment) may be very poor in sector 0 (those who receive no treatment). Hence the bias in OLS for the parameter treatment on the treated may be negative ($E(Y_0 \mid X, D = 1) - E(Y_0 \mid X, D = 0) < 0$).

The differences among the policy relevant treatment effects, the conventional treatment effects and the OLS estimand are illustrated in figure 8A and table 3 for the generalized Roy model example. As is evident from table 3, it is not at all clear that the instrumental variable estimator, with instruments that satisfy classical properties, performs better than nonparametric OLS in identifying the policy relevant treatment effect in this example. While IV eliminates the term in braces in (16), it reweights the MTE differently from what might be desired for many policy analyses.

If there is no selection on unobserved variables conditional on covariates, $U_D \perp (Y_0, Y_1) \mid X$, then $E(U_1 \mid X, U_D) = E(U_1 \mid X) = 0$ and $E(U_0 \mid X, U_D) = E(U_0 \mid X) = 0$ so that the OLS weights are unity and OLS identifies both ATE and the parameter treatment on the treated (TT), which are the same under this assumption. This condition is an implication of matching condition (M-1). Given the assumed conditional independence in terms of $X$, we can identify ATE and TT without use of any instrument $Z$ satisfying assumptions (A-1)–(A-2). If there is such a $Z$, the conditional independence condition implies under (A-1)–(A-5) that $E(Y \mid X, P(Z) = p)$ is linear in $p$. The conditional independence assumption invoked in the method of matching has come into
widespread use for much the same reason that OLS has come into widespread use. It is easy to implement with modern software and makes little demands of the data because it assumes the existence of $X$ variables that satisfy the conditional independence assumptions. The crucial conditional independence assumption is not testable. As I note below, additional assumptions on the $X$ are required to test the validity of the matching assumptions.

If the sole interest is to identify treatment on the treated, $\Delta^{TT}$, it is apparent from representation (16) that one can weaken (M-1) to

$$(M-1)' \quad Y_0 \perp D \mid X.$$  

This is possible because we know $E(Y_1 \mid X, D = 1)$ from data on outcomes of the treated and only need to construct $E(Y_0 \mid X, D = 1)$. In this case MTE is not restricted to be flat in $u_D$ and all treatment parameters are not the same. A straightforward implication of $(M-1)'$ in the Roy model, where selection is made solely on the gain, is that persons must sort into treatment status positively in terms of levels of $Y_1$. I now consider more generally the implications of assuming mean independence of the errors rather than full independence.

### 3.2 Matching and MTE Using Mean Independence Conditions

To identify all mean treatment parameters, one can weaken the assumption (M-1) to the condition that $Y_1$ and $Y_0$ are mean independent of $D$ conditional on $X$. However, $(Y_0, Y_1)$ will be mean independent of $D$ conditional on $X$ without $U_D$ being independent of $Y_0, Y_1$ conditional on $X$ only if fortuitous balancing occurs, with regions of positive dependence of $(Y_0, Y_1)$ on $U_D$ and regions of negative dependence of $(Y_0, Y_1)$ on $U_D$ just exactly offsetting each other. Such balancing is not generic in the Roy model and in the generalized Roy model.

In particular, assume $Y_j = \mu_j(X) + U_j$ for $j = 0, 1$ and further assume that $D = 1(Y_1 - Y_0 \geq C(Z) + U_C)$. Let $V = U_C - (U_1 - U_0)$. Assume $(U_0, U_1, V) \perp (X, Z)$. Then if $V \perp (U_1 - U_0)$, and $U_C$ has a log concave density, then $E(Y_1 - Y_0 \mid X, V = v)$ is decreasing in $v$, $\Delta^{TT}(x) > \Delta^{ATE}(x)$, and the matching conditions do not hold. If $V \perp (U_1 - U_0)$ but $V$ does not have a log concave density, then it is still the case that $(U_1 - U_0, V)$ is negative quadrant dependent. One can show that
\((U_1 - U_0, V)\) being negative quadrant dependent implies that \(\Delta^{TT}(x) > \Delta^{ATE}(x)\), and thus again that the matching conditions cannot hold. I now consider a more general analysis.

Suppose that we assume selection model (12) so that \(D = 1[P(Z) \geq U_D]\), where \(Z\) is independent of \((Y_0, Y_1)\) conditional on \(X\), where \(U_D = F_{V|X}(V)\) and \(P(Z) = F_{V|X}(\mu_D(Z))\). Consider the weaker mean independence assumptions in place of assumption (M-1):

\[
(M-3) \quad E(Y_1|X, D) = E(Y_1|X), \quad E(Y_0|X, D) = E(Y_0|X).
\]

This assumption is all that is needed to identify the mean treatment parameters because under it

\[
E(Y|X = x, Z = z, D = 1) = E(Y_1|X = x, Z = z, D = 1) = E(Y_1|X = x)
\]

and

\[
E(Y|X = x, Z = z, D = 0) = E(Y_0|X = x, Z = z, D = 0) = E(Y_0|X = x).
\]

Thus one can identify all the mean treatment parameters over the support that satisfies (M-2).

Recalling that \(\Delta = Y_1 - Y_0\), (M-3) implies in terms of \(U_D\) that

\[
E(\Delta|X = x, Z = z, U_D \leq P(z)) = E(\Delta|X = x)
\]

\[
\Leftrightarrow E(\Delta^{MTE}(X, U_D)|X = x, U_D \leq P(z)) = E(\Delta|X = x),
\]

and hence

\[
E(\Delta^{MTE}(X, U_D)|X = x, U_D \leq P(z)) = E(\Delta^{MTE}(X, U_D)|X = x, U_D > P(z)).
\]

If the support of \(P(Z)\) is the full unit interval conditional on \(X = x\), then \(\Delta^{MTE}(X, U_D) = E(\Delta|X = x)\) for all \(U_D\). If the support of \(P(Z)\) is a proper subset of the full unit interval, then generically (M-3) will hold only if \(\Delta^{MTE}(X, U_D) = E(\Delta|X = x)\) for all \(U_D\), though positive and negative parts could balance out for any particular value of \(X\).
To see this, note that

\[
E_Z \left( E(\Delta^{\text{MTE}}(X, U_D)|X = x, U_D \leq P(z))|X = x, D = 1 \right) = E_Z(E(\Delta^{\text{MTE}}(X, U_D)|X = x, U_D > P(z))|X = x, D = 0). 
\]

Working with \( V = F_{-1}^{-1}(U_D) \), suppose that \( D = 1[\mu_D(Z, V) \geq 0] \). Let \( \Omega(z) = \{v : \mu_D(z, v) \geq 0\} \). Then (M-3) implies that

\[
E(\Delta^{\text{MTE}}(X, V)|X = x, V \in \Omega(z)) = E(\Delta^{\text{MTE}}(X, V)|X = x, V \in (\Omega(z))^c)
\]

so one expects that generically under assumption (M-3) one obtains a flat MTE in terms of \( V = F_{-1}^{-1}(U_D) \).\textsuperscript{37} Matching assumes a flat MTE, i.e. that marginal returns conditional on \( X \) and \( V \) do not depend on \( V \) (alternatively, that marginal returns do not depend on \( U_D \) given \( X \)).

I previously noted that IV and matching invoke very different assumptions. Matching requires no exclusion restrictions whereas IV is based on the existence of exclusion restrictions. Superficially, we can bridge these literatures by invoking matching with an exclusion condition: \((Y_0, Y_1) \not\perp D | X\) but \((Y_0, Y_1) \perp D | X, Z\). This looks like an IV condition, but it is not.

Heckman and Vytlacil (2007b, appendix L) explore the relationship between matching with exclusion and IV and demonstrate a fundamental contradiction between the two identifying conditions. For an additively separable representation of the outcome equations \( U_1 = Y_1 - E(Y_1|X) \) and \( U_0 = Y_0 - E(Y_0|X) \), if \((U_0, U_1)\) is mean independent of \( D \) conditional on \((X, Z)\), as required by IV, but \((U_0, U_1)\) is not mean independent of \( D \) conditional on \( X \) alone, then \( U_0 \) is dependent on \( Z \) conditional on \( X \), contrary to all assumptions used to justify instrumental variables. I next consider how to implement matching.

\textsuperscript{37}Heckman and Vytlacil (2007b, appendix K) conduct a parallel analysis for the nonseparable choice model and obtain similar conditions.
3.3 Implementing the Method of Matching

I draw on Heckman, Ichimura, Smith, and Todd (1998) and Heckman, LaLonde, and Smith (1999) to describe the mechanics of matching. Todd (2006a,b) presents a comprehensive treatment of the main issues and a guide to software.

To operationalize the method of matching, I assume two samples: “t” for treatment and “c” for comparison group. Treatment group members have \( D = 1 \) and control group members have \( D = 0 \). Unless otherwise noted, I assume that observations are statistically independent within and across groups. Simple matching methods are based on the following idea. For each person \( i \) in the treatment group, I find some group of “comparable” persons. The same individual may be in both treated and control groups if that person is treated at one time and untreated at another. I denote outcomes for person \( i \) in the treatment group by \( Y_{ti} \) and I match these outcomes to the outcomes of a subsample of persons in the comparison group to estimate a treatment effect. In principle, I can use a different subsample as a comparison group for each person.

In practice, one can construct matches on the basis of a neighborhood \( \xi(X_i) \), where \( X_i \) is a vector of characteristics for person \( i \). Neighbors to treated person \( i \) are persons in the companion sample whose characteristics are in neighborhood \( \xi(X_i) \). Suppose that there are \( N_c \) persons in the comparison sample and \( N_t \) in the treatment sample. Thus the persons in the comparison sample who are neighbors to \( i \), are persons \( j \) for whom \( X_j \in \xi(X_i) \), i.e., the set of persons \( A_i = \{j \mid X_j \in \xi(X_i)\} \). Let \( W(i, j) \) be the weight placed on observation \( j \) in forming a comparison with observation \( i \) and further assume that the weights sum to one, \( \sum_{j=1}^{N_c} W(i, j) = 1 \), and that \( 0 \leq W(i, j) \leq 1 \). Form a weighted comparison group mean for person \( i \), given by

\[
\bar{Y}_c^i = \sum_{j=1}^{N_c} W(i, j) Y_c^j.
\]

(17)

The estimated treatment effect for person \( i \) is \( Y_i - \bar{Y}_c^i \). This selects a set of comparison group members associated with \( i \) and the mean of their outcomes. Unlike IV or the control function approach, the method of matching identifies counterfactuals for each treated member.

Heckman, Ichimura, and Todd (1997) and Heckman, LaLonde, and Smith (1999) survey a
variety of alternative matching schemes proposed in the literature. Todd (2006a,b) provides a comprehensive survey. I briefly consider two widely-used methods. The nearest-neighbor matching estimator defines $A_i$ such that only one $j$ is selected so that it is closest to $X_i$ in some metric:

$$A_i = \{ j \mid \min_{j \in [1,\ldots,N_c]} \| X_i - X_j \| \},$$

where “$\| \|” is a metric measuring distance in the $X$ characteristics space. The Mahalanobis metric is one widely used metric for implementing the nearest neighbor matching estimator. This metric defines neighborhoods for $i$ as

$$\| \| = (X_i - X_j)' \sum_c^{-1} (X_i - X_j),$$

where $\sum_c$ is the covariance matrix in the comparison sample. The weighting scheme for the nearest neighbor matching estimator is

$$W(i, j) = \begin{cases} 1 & \text{if } j \in A_i, \\ 0 & \text{otherwise}. \end{cases}$$

The nearest neighbor in the metric “$\| \|$” is used in the match. A version of nearest-neighbor matching, called “caliper” matching (Cochran and Rubin, 1973), makes matches to person $i$ only if

$$\| X_i - X_j \| < \varepsilon,$$

where $\varepsilon$ is a pre-specified tolerance. Otherwise person $i$ is bypassed and no match is made to him or her.

Kernel matching uses the entire comparison sample, so that $A_i = \{1,\ldots,N_c\}$, and sets

$$W(i, j) = \frac{K(X_j - X_i)}{\sum_{j=1}^{N_c} K(X_j - X_i)},$$

where $K$ is a kernel.\textsuperscript{38} Kernels are typically a standard distribution function such as the normal

\textsuperscript{38}See, e.g., Härdle (1990) or Ichimura and Todd (2007) for a discussion of kernels and choices of bandwidths.
cumulative distribution function. Kernel matching is a smooth method that reuses and weights the comparison group sample observations differently for each person \(i\) in the treatment group with a different \(X_i\). Kernel matching can be defined pointwise at each sample point \(X_i\) or for broader intervals.

For example, the impact of treatment on the treated can be estimated by forming the mean difference across the \(i\):

\[
\hat{\Delta}^{TT} = \frac{1}{N_t} \sum_{i=1}^{N_t} (Y_i^t - \tilde{Y}_i^c) = \frac{1}{N_t} \sum_{i=1}^{N_t} (Y_i^t - \sum_{j=1}^{N_c} W(i, j)Y_j^c).
\]  

(18)

One can define this mean for various subsets of the treatment sample defined in various ways. More efficient estimators weight the observations accounting for the variance (Abadie and Imbens, 2006; Heckman, 1998; Heckman, Ichimura, and Todd, 1997, 1998; Hirano, Imbens, and Ridder, 2003).\(^{39}\)

Matching assumes that conditioning on \(X\) eliminates selection bias. The method requires no functional form assumptions for outcome equations. If, however, a functional form assumption is maintained, as in the econometric procedure proposed by Barnow, Cain, and Goldberger (1980), it is possible to implement the matching assumption using standard regression analysis. Suppose, for example, that \(Y_0\) is linearly related to observables \(X\) and an unobservable \(U_0\), so that

\[
E(Y_0 | X, D = 0) = X\beta + E(U_0 | X, D = 0),
\]

and

\[
E(U_0 | X, D = 0) = E(U_0 | X)
\]

is linear in \(X\) (\(E(U | X) = \phi X\)). Under these assumptions, controlling for \(X\) via linear regression allows one to identify \(E(Y_0 | X, D = 1)\) from the data on nonparticipants. Setting \(X = Q\), this approach justifies OLS equation (Q-3).\(^{40}\) Such functional form assumptions are not strictly

\(^{39}\)Regression-adjusted matching, proposed by Rubin (1979) and clarified in Heckman, Ichimura, and Todd (1997, 1998), uses regression-adjusted \(Y_i\), denoted by \(\tau(Y_i) = Y_i - X_i\beta\), in place of \(Y_i\) in the preceding calculations. (See the cited papers for the econometric details of the procedure).

\(^{40}\)In equation (Q-3), this approach shows that \(\pi\) combines the estimate \(U_0 - Q\) effect with the causal effect of \(Q\) on \(Y\).
required to implement the method of matching. Moreover, in practice, users of the method of Barnow, Cain, and Goldberger (1980) do not impose the common support condition (M-2) for the distribution of $X$ when generating estimates of the treatment effect. The distribution of $X$ may be very different in the treatment group ($D = 1$) and comparison group ($D = 0$) samples, so that comparability is only achieved by imposing linearity in the parameters and extrapolating over different regions.

One advantage of the method of Barnow, Cain, and Goldberger (1980) is that it uses data parsimoniously. If the $X$ are high dimensional, the number of observations in each cell when matching can get very small.

Another solution to this problem that reduces the dimension of the matching problem without imposing arbitrary linearity assumptions is based on the probability of participation or the “propensity score,” $P(X) = \Pr(D = 1 \mid X)$. Rosenbaum and Rubin (1983) demonstrate that under assumptions (M-1) and (M-2),

\[(Y_0, Y_1) \perp D \mid P(X) \text{ for } X \in \chi_c, \quad (19)\]

for some set $\chi_c$, where it is assumed that (M-2) holds in the set. Conditioning either on $P(X)$ or on $X$ produces conditional independence.\(^{41}\)

Conditioning on $P(X)$ reduces the dimension of the matching problem down to matching on the scalar $P(X)$. The analysis of Rosenbaum and Rubin (1983) assumes that $P(X)$ is known rather than estimated. Heckman, Ichimura, and Todd (1998), Hahn (1998), and Hirano, Imbens, and Ridder (2003) present the asymptotic distribution theory for the kernel matching estimator in the cases in which $P(X)$ is known and in which it is estimated both parametrically and nonparametrically.

Conditioning on $P$ identifies all treatment parameters but as has been shown, it imposes the assumption of a flat MTE. Marginal returns and average returns are the same. A consequence of

\(^{41}\)Their analysis is generalized to a multiple treatment setting in Lechner (2001) and Imbens (2003).
(19) is that

\[
E(Y_1 \mid D = 0, P(X)) = E(Y_1 \mid D = 1, P(X)) = E(Y_1 \mid P(X)),
\]

\[
E(Y_0 \mid D = 1, P(X)) = E(Y_0 \mid D = 0, P(X)) = E(Y_0 \mid P(X)).
\]

Support condition (M-2) has the unattractive feature that if the analyst has too much information about the decision of who takes treatment, so that \( P(X) = 1 \) or \( 0 \), the method breaks down at such values of \( X \) because people cannot be compared at a common \( X \). The method of matching assumes that, given \( X \), some unspecified randomization in the economic environment allocates people to treatment. This produces assumption (Q-5) in the OLS example. The fact that the cases \( P(X) = 1 \) and \( P(X) = 0 \) must be eliminated suggests that methods for choosing \( X \) based on the fit of the model to data on \( D \) are potentially problematic, as I discuss below.

Offsetting these disadvantages, the method of matching with a known conditioning set that produces condition (M-2) does not require separability of outcome or choice equations, exogeneity of conditioning variables, exclusion restrictions, or adoption of specific functional forms of outcome equations. Such features are commonly used in conventional selection (control function) methods and conventional applications of IV although recent work in semiparametric estimation relaxes these assumptions. As noted in section 3.2, the method of matching does not strictly require (M-1). One can get by with weaker mean independence assumptions (M-3) in the place of the stronger conditions (M-1). However, if (M-3) is invoked, the assumption that one can replace \( X \) by \( P(X) \) does not follow from the analysis of Rosenbaum and Rubin (1983), and is an additional new assumption.

Methods for implementing matching are provided in Heckman, Ichimura, Smith, and Todd (1998) and are discussed extensively in Heckman, LaLonde, and Smith (1999). See Todd (1999, 2006a,b) for software and extensive discussion of the mechanics of matching. I now contrast the identifying assumptions used in the method of control functions with those used in matching.
3.3.1 Comparing Matching and Control Functions Approaches

The method of matching eliminates the dependence between \((Y_0, Y_1)\) and \(D\), \((Y_0, Y_1) \not\perp D\), by assuming access to conditioning variables \(X\) such that (M-1) is satisfied: \((Y_0, Y_1) \not\perp D \mid X\). By conditioning on observables, one can identify the distributions of \(Y_0\) and \(Y_1\) over the support of \(X\) satisfying (M-2).

Other methods model the dependence that gives rise to the spurious relationship and in this way attempt to eliminate it. IV involves exclusion and a different type of conditional independence, \((Y_0, Y_1) \not\perp Z \mid X\), as well as a rank condition (\(\Pr(D = 1 \mid X, Z)\) depends on \(Z\)). The instrument \(Z\) plays the role of the implicit randomization used in matching by allocating people to treatment status in a way that does not depend on \((Y_0, Y_1)\). I have already established that matching and IV make very different assumptions. Thus, in general, a matching assumption that \((Y_0, Y_1) \not\perp D \mid X, Z\) neither implies nor is implied by \((Y_0, Y_1) \not\perp Z \mid X\). One special case where they are equivalent is when treatment status is assigned by randomization with full compliance (letting \(\xi = 1\) denote assignment to treatment, \(\xi = 1 \Rightarrow A = 1\) and \(\xi = 0 \Rightarrow A = 0\)) and \(Z = \xi\), so that the instrument is the assignment mechanism. \(A = 1\) if the person actually receives treatment, and \(A = 0\) otherwise.

The method of control functions explicitly models the dependence between \((Y_0, Y_1)\) and \(D\) and attempts to eliminate it. Matzkin (2007) provides a comprehensive review of these methods. I previously presented a summary of some of the general principles underlying the method of control functions, the method of control variates, replacement functions, and proxy approaches as they apply to the selection problem that underlies the evaluation problem studied in this paper. All of these methods attempt to eliminate the \(\theta\) in (U-1) that produces the dependence captured in (U-2).

This section relates matching to the form of the control function introduced in Heckman (1980) and Heckman and Robb (1985a, 1986a). I analyze conditional means because of their familiarity. Using the fact that \(E(1(Y \leq y) \mid X) = F(y \mid X)\) the analysis applies to marginal distributions as well.

Thus I work with conditional expectations of \((Y_0, Y_1)\) given \((X, Z, D)\), where \(Z\) is assumed to
include at least one variable not in $X$. Conventional applications of the control function method assume additive separability, which is not required in matching. Strictly speaking, additive separability is not required in the application of control functions either.\textsuperscript{42} What is required is a model relating the outcome unobservables to the observables and the unobservables in the choice of treatment equation. Various assumptions give operational content to (U-1).

For the additively separable case (7), the control function for mean outcomes models the conditional expectations of $Y_1$ and $Y_0$ given $X$, $Z$, and $D$ as

\[
E(Y_1|Z, X, D = 1) = \mu_1(X) + E(U_1|Z, X, D = 1)
\]

\[
E(Y_0|Z, X, D = 0) = \mu_0(X) + E(U_0|Z, X, D = 0).
\]

In the traditional method of control functions, the analyst models $E(U_1|Z, X, D = 1)$ and $E(U_0|Z, X, D = 0)$. If these functions can be independently varied against $\mu_1(X)$ and $\mu_0(X)$ respectively, one can identify $\mu_1(X)$ and $\mu_0(X)$ up to constant terms.\textsuperscript{43} It is not required that $X$ or $Z$ be stochastically independent of $U_1$ or $U_0$, although conventional methods often assume this.

Assume that $(U_1, U_0, V) \perp \perp (X, Z)$ and adopt equation (12) as the treatment choice model augmented so that $X$ and $Z$ are determinants of treatment choice, using $V$ as the latent variable that generates $D$ given $X, Z$: $D = 1(\mu_D(Z) > V)$. Let $U_D = F_{V|X}(V)$ and $P(Z) = F_{V|X}(\mu_D(Z))$. In this notation, the control functions are

\[
E(U_1|Z, D = 1) = E(U_1|\mu_D(Z) \geq V) = E(U_1 | P(Z) \geq U_D) = K_1(P(Z)) \quad \text{and}
\]

\[
E(U_0|Z, D = 0) = E(U_0|\mu_D(Z) < V) = E(U_0 | P(Z) < U_D) = K_0(P(Z)),
\]

so the control function only depends on the propensity score $P(Z)$. The key assumption needed\textsuperscript{42}Examples of nonseparable selection models are found in Cameron and Heckman (1998). See also Altonji and Matzkin (2005) and Matzkin (2007).

\textsuperscript{43}Heckman and Robb (1985a, 1986a) introduce this general formulation of control functions. The identifiability requires that the members of the pairs $(\mu_1(X), E(U_1|X, Z, D = 1))$ and $(\mu_0(X), E(U_0|X, Z, D = 0))$ be variation-free so that they can be independently varied against each other.
to represent the control function solely as a function of $P(Z)$ is

$$(U_1, U_0, V) \perp X, Z. \quad \text{(CF-1)}$$

This assumption is not strictly required. Under this condition

$$E(Y_1|Z, X, D = 1) = \mu_1(X) + K_1(P(Z)),$$
$$E(Y_0|Z, X, D = 0) = \mu_0(X) + K_0(P(Z)),$$

with $\lim_{P \to 1} K_1(P) = 0$ and $\lim_{P \to 0} K_0(P) = 0$. It is assumed that $Z$ can be independently varied for all $X$, and the limits are obtained by changing $Z$ while holding $X$ fixed.\(^\text{44}\) These limit results state that when the values of $X, Z$ are such that the probability of being in a sample ($D = 1$ or $D = 0$, respectively) is 1, there is no selection bias and one can separate out $\mu_1(X)$ from $K_1(P(Z))$ and $\mu_0(X)$ from $K_0(P(Z))$. This is the same identification at infinity condition that is required to identify ATE and TT in IV for models with heterogeneous responses.\(^\text{45, 46}\)

Unlike the method of matching based on (M-1), the method of control functions allows the marginal treatment effect to be different from the average treatment effect and from the conditional effect of treatment on the treated. Although conventional practice has been to derive the functional forms of $K_0(P)$ and $K_1(P)$ by making distributional assumptions about $(U_0, U_1, V)$ such as normality or other conventional distributional assumptions, this is not an intrinsic feature of the method and there are many nonnormal and semiparametric versions of this method. See Powell (1994) for a survey.

In its semiparametric implementation, the method of control functions requires an exclusion restriction (a variable in $Z$ not in $X$) to achieve nonparametric identification.\(^\text{47}\) Without any functional-form assumptions one cannot rule out a worst case analysis where, for example, if $X =$

\(^{\text{44}}\) More precisely, I assume that $\text{Supp}(Z|X) = \text{Supp}(Z)$ and that limit sets of $Z, Z_0,$ and $Z_1$ exist so that as $Z \to Z_0, P(Z, X) \to 0,$ and as $Z \to Z_1, P(Z, X) \to 1.$

\(^{\text{45}}\) One needs identification at infinity to obtain ATE and TT. This is a general feature of any evaluation model with general heterogeneity.

\(^{\text{46}}\) One can approximate the $K_1(P)$ and $K_0(P)$ terms by polynomials in $P$ (see Heckman, 1980; Heckman and Hotz, 1989; Heckman and Robb, 1985a, 1986a). Ahn and Powell (1993) and Powell (1994) develop methods for eliminating $K_1(P(Z))$ and $K_0(P(Z))$ by differentiating.

\(^{\text{47}}\) No exclusion is required for many common functional forms for the distributions of unobservables.
$Z$, then $K_1 (P(X)) = \alpha \mu (X)$ where $\alpha$ is a scalar. In this situation, there is perfect collinearity between the control function and the conditional mean of the outcome equation, and it is impossible to separately identify either.\footnote{Clearly $K_1 (P(X))$ and $\mu (P)$ cannot be independently varied in this case. Olsen (1980) develops the worst case scenario.} Even though this case is not generic, it is possible. The method of matching does not require an exclusion restriction, but at the cost of ruling out essential heterogeneity. In the general case, the method of control functions requires that in certain limit sets of $Z$, $P(Z) = 1$ and $P(Z) = 0$ in order to achieve full nonparametric identification.\footnote{Symmetry of the errors can be used in place of the appeal to limit sets that put $P(Z) = 0$ or $P(Z) = 1$. See Chen (1999).} The conventional method of matching does not invoke such limit set arguments.

All methods of evaluation, including matching and control functions, require that treatment parameters be defined on a common support that is the intersection of the supports of $X$ given $D = 1$ and $X$ given $D = 0$: $\text{Supp}(X|D = 1) \cap \text{Supp}(X|D = 0)$. This is the requirement for any estimator that seeks to identify treatment effects by comparing samples of treated persons with samples of untreated persons. It is a great advantage of nonparametric methods that they impose this condition.

In this version of the method of control functions, $P(Z)$ is a conditioning variable used to predict $U_1$ conditional on $D$ and $U_0$ conditional on $D$. In the method of matching, it is used as a conditioning variable to eliminate the stochastic independence between $(U_0, U_1)$ and $D$. In the method of LATE or LIV, $P(Z)$ is used as an instrument. In the method of control functions, as conventionally applied, $(U_0, U_1) \perp \perp (X, Z)$, but this assumption is not intrinsic to the method.\footnote{Relaxing it, however, requires that the analyst model the dependence of the unobservables on the observables and that certain variation-free conditions are satisfied. (See Heckman and Robb, 1985a).} This assumption plays no role in matching if the correct conditioning set is known.\footnote{That is, a conditioning set that satisfies (M-1) and (M-2).} However, as noted below, exogeneity plays a key role in devising algorithms to select the conditioning variables. In addition, as developed in Heckman and Vytlacil (2005), exogeneity is helpful in making out-of-sample forecasts. The method of control functions does not require that $(U_0, U_1) \perp \perp D|(X, Z)$, which is a central requirement of matching. Equivalently, the method
of control functions does not require

\[(U_0, U_1) \perp \perp V| (X, Z), \quad \text{or that} \quad (U_0, U_1) \perp \perp V| X\]

whereas matching does and typically equates X and Z. Thus matching assumes access to a richer set of conditioning variables than is assumed in the method of control functions.

The method of control functions allows for outcome unobservables to be dependent on D even after conditioning on \((X, Z)\), and it models this dependence. The method of matching assumes no such D dependence. Thus in this regard, and maintaining all of the assumptions invoked for control functions in this section, matching is a special case of the method of control functions in which under assumptions (M-1) and (M-2),

\[
E(U_1 | X, D = 1) = E(U_1 | X) \\
E(U_0 | X, D = 0) = E(U_0 | X).
\]

In the method of control functions, in the case where \((X, Z) \perp \perp (U_0, U_1, V)\), where the Z can include some or all of the elements of X, the conditional expectation of Y given X, Z, D is

[20]

\[
E(Y | X, Z, D) = E(Y_1 | X, Z, D = 1) D + E(Y_0 | X, Z, D = 0) (1 - D) \\
= \mu_0(X) + [\mu_1(X) - \mu_0(X)] D \\
+ E(U_1 | P(Z), D = 1) D + E(U_0 | P(Z), D = 0) (1 - D) \\
= \mu_0(X) + K_0(P(Z)) + [\mu_1(X) - \mu_0(X) + K_1(P(Z)) - K_0(P(Z))] D.
\]

The coefficient on D in the final equation combines \(\mu_1(X) - \mu_0(X)\) with \(K_1(P(Z)) - K_0(P(Z))\). It does not correspond to any treatment effect. To identify \(\mu_1(X) - \mu_0(X)\), one must isolate it from \(K_1(P(Z)) - K_0(P(Z))\).

Under assumptions (M-1) and (M-2) of the method of matching, the conditional expectation

\[52\]See Aakvik, Heckman, and Vytlacil (2005), Carneiro, Hansen, and Heckman (2003) and Cunha, Heckman, and Navarro (2005) for a generalization of matching that allows for selection on unobservables by imposing a factor structure on the errors and estimating the distribution of the unobserved factors. These methods are discussed in Abbring and Heckman (2007).
of $Y$ conditional on $P(X)$ and $D$ is

$$E(Y|P(X), D) = \mu_0(P(X)) + E(U_0|P(X))$$

$$+ \left[ (\mu_1(P(X)) - \mu_0(P(X))) + E(U_1|P(X)) - E(U_0|P(X)) \right] D.$$  

The coefficient on $D$ in this expression is now interpretable and is the average treatment effect. If one assumes that $(U_0, U_1) \perp \perp X$, which is not strictly required, one obtains a familiar representation

$$E(Y|P(X), D) = \mu_0(P(X)) + [\mu_1(P(X)) - \mu_0(P(X))] D,$$  

(22)

since $E(U_1|P(X)) = E(U_0|P(X)) = 0$. A parallel derivation can be made conditioning on $X$ instead of $P(X)$.

Under the assumptions that justify matching, treatment effects ATE or TT (conditional on $P(X)$) are identified from the coefficient on $D$ in either (21) or (22). Condition (M-2) guarantees that $D$ is not perfectly predictable by $X$ (or $P(X)$), so the variation in $D$ identifies the treatment parameter.

The coefficient on $D$ in equation (20) for the more general control function model does not correspond to any treatment parameter, whereas the coefficients on $D$ in equations (21) and (22) correspond to treatment parameters under the assumptions of the matching model. Under assumption (CF-1), $\mu_1(P(X)) - \mu_0(P(X)) = \text{ATE}$ and $\text{ATE} = \text{TT} = \text{MTE}$, so the method of matching identifies all of the (conditional on $P(X)$) mean treatment parameters.\footnote{This result also holds even if (CF-1) is not satisfied because $(U_0, U_1) \not\perp \not\perp X$. In this case, the treatment effects include the term $E(U_1 | P(X)) - E(U_0 | P(X))$.}

Under the assumptions justifying matching, when means of $Y_1$ and $Y_0$ are the parameters of interest, and $X$ satisfies (M-1) and (M-2), the bias terms vanish. They do not vanish in the more general case considered by the method of control functions. This is the mathematical counterpart of the randomization implicit in matching: conditional on $X$ or $P(X)$, $(U_1, U_0)$ are random with respect to $D$. The method of control functions allows these error terms to be nonrandom with respect to $D$ and models the dependence. In the absence of functional form assumptions,
it requires an exclusion restriction (a variable in \( Z \) not in \( X \)) to separate out \( K_0(P(Z)) \) from the coefficient on \( D \). Matching produces identification without exclusion restrictions whereas identification with exclusion restrictions is a central feature of the control function method in the absence of functional form assumptions.

The fact that the control function approach allows for more general dependencies among the unobservables and the conditioning variables than the matching approach allows is implicitly recognized in the work of Rosenbaum (1995) and Robins (1997). Their “sensitivity analyses” for matching when there are unobserved conditioning variables are, in their essence, sensitivity analyses using control functions.\(^{54}\) Aakvik, Heckman, and Vytlacil (2005), Carneiro, Hansen, and Heckman (2003) and Cunha, Heckman, and Navarro (2005) explicitly model the relationship between matching and selection models using factor structure models, treating the omitted conditioning variables as unobserved factors and estimating their distribution. Abbring and Heckman (2007) discuss this work.

3.4 Comparing Matching and Classical Control Function Methods for a Generalized Roy Model

Figure 6 shows the contrast between the shape of the MTE and the OLS matching estimand as a function of \( p \) for the extended Roy model. The MTE\((p)\) shows its typical declining shape associated with diminishing returns, and the assumptions justifying matching are violated. Matching attempts to impose a flat MTE\((p)\) and therefore flattens the estimated MTE\((p)\) compared to its true value. It understates marginal returns at low levels of \( p \) (associated with unobservables that make it likely to participate in treatment) and overstates marginal returns at high levels of \( p \).

To further illustrate the bias in matching and how the control function eliminates it, I perform sensitivity analyses under different assumptions about the parameters of the underlying selection model. In particular, I assume that the data are generated by the model of equations 10 and 11,

\(^{54}\)See also Vijverberg (1993) who does such a sensitivity analysis in a parametric selection model with an unidentified parameter.
where \( \mu_D(Z) = Z\gamma, \mu_0(X) = \mu_0, \mu_1(X) = \mu_1, \) and

\[
(U_1, U_0, V)' \sim N(0, \Sigma) \\
\text{corr}(U_j, V) = \rho_{jV} \\
\text{Var}(U_j) = \sigma_{jV}^2, \ j = \{0, 1\}.
\]

I assume in this section that \( D = 1[\mu_D(Z) + V > 0], \) in conformity with the examples presented in Heckman and Navarro (2004), on which we build. This reformulation of choice model (12) entails a simple change in the sign of \( V. \) I assume that \( Z \perp \perp (U_1, U_0, V). \) Using the selection formulae derived in appendix A, we can write the biases conditional on \( P(Z) = p \) as

\[
\text{Bias TT } (Z = z) = \text{Bias TT } (P(Z) = p) = \sigma_0 \rho_{0V} M(p) \\
\text{Bias ATE } (Z = z) = \text{Bias ATE } (P(Z) = p) = M(p) [\sigma_1 \rho_{1V} (1 - p) + \sigma_0 \rho_{0V} p]
\]

where \( M(p) = \frac{\phi(\Phi^{-1}(1-p))}{p(1-p)}, \) \( \phi(\cdot) \) and \( \Phi(\cdot) \) are the pdf and cdf of a standard normal random variable and the propensity score \( P(z) \) is evaluated at \( P(z) = p. \) I assume that \( \mu_1 = \mu_0 \) so that the true average treatment effect is zero.

I simulate the mean bias for TT (table 4) and ATE (table 5) for different values of the \( \rho_{jV} \) and \( \sigma_j. \) The results in the tables show that, as one lets the variances of the outcome equations grow, the value of the mean bias that we obtain can become substantial. With larger correlations between the outcomes and the unobservables generating choices, come larger biases. These tables demonstrate the greater generality of the control function approach, which models the bias rather than assuming it away by conditioning. Even if the correlation between the observables and the unobservables \( (\rho_{jV}) \) is small, so that one might think that selection on unobservables is relatively unimportant, we still obtain substantial biases if we do not control for relevant omitted conditioning variables. Only for special values of the parameters can one avoid bias by matching. These examples also demonstrate that sensitivity analyses can be conducted for analysis based on control function methods even when they are not fully identified. Vijverberg (1993) provides an example.
### 3.5 The Informational Requirements of Matching and the Bias When They Are Not Satisfied

In this section I present some examples of when matching “works” and when it breaks down. This section extends the analysis of Heckman and Navarro (2004). In particular, I show how matching on some of the relevant information but not all can make the bias using matching worse for standard treatment parameters.

Section 1 discussed informational asymmetries between the econometrician and the agents whose behavior they are analyzing. The method of matching assumes that the econometrician has access to and uses all of the relevant information in the precise sense defined there. That means that the $X$ that guarantees conditional independence (M-1) is available and is used. The concept of relevant information is a delicate one and it is difficult to find the true conditioning set.

Assume that the economic model generating the data is a generalized Roy model of the form

$$D^* = Z \gamma + V$$

where

$$Z \perp \perp V$$

and

$$V = \alpha_{V1} f_1 + \alpha_{V2} f_2 + \epsilon_V$$

$$D = \begin{cases} 
1 & \text{if } D^* \geq 0 \\
0 & \text{otherwise}
\end{cases}$$

and

$$Y_1 = \mu_1 + U_1 \quad \text{where } U_1 = \alpha_{11} f_1 + \alpha_{12} f_2 + \epsilon_1,$$

$$Y_0 = \mu_0 + U_0 \quad \text{where } U_0 = \alpha_{01} f_1 + \alpha_{02} f_2 + \epsilon_0.$$}

Observe that contrary to the analysis throughout this paper we add $V$ and do not subtract it in the decision equation. This is the familiar representation. By a change in sign in $V$ one can go back and forth between the specification used in this section and the specification used in other
sections of the paper.

In this specification, \((f_1, f_2, \epsilon_V, \epsilon_1, \epsilon_0)\) are assumed to be mean zero random variables that are mutually independent of each other and \(Z\) so that all the correlation among the elements of \((U_0, U_1, V)\) is captured by \(f = (f_1, f_2)\). Models that take this form are known as factor models and have been applied in the context of selection models by Aakvik, Heckman, and Vytlacil (2005), Carneiro, Hansen, and Heckman (2001, 2003), and Hansen, Heckman, and Mullen (2004), among others. I keep implicit any dependence on \(X\) which may be general.

Generically, the minimal relevant information for this model when the factor loadings are not zero \((\alpha_{ij} \neq 0)\) is, for general values of the factor loadings,

\[
I_R = \{f_1, f_2\}.
\]

Recall that I assume independence between \(Z\) and all error terms. If the econometrician has access to \(I_R\) and uses it, (M-1) is satisfied conditional on \(I_R\). Note that \(I_R\) plays the role of \(\theta\) in (U-1). In the case where the economist knows \(I_R\), the economist’s information set \(\sigma(I_E)\) contains the relevant information \((\sigma(I_E) \supsetneq \sigma(I_R))\).

The agent’s information set may include different variables. If one assumes that \(\epsilon_0, \epsilon_1\) are shocks to outcomes not known to the agent at the time treatment decisions are made, but the agent knows all other aspects of the model, the agent’s information is

\[
I_A = \{f_1, f_2, Z, \epsilon_V\}.
\]

Under perfect certainty, the agent’s information set includes \(\epsilon_1\) and \(\epsilon_0\):

\[
I_A = \{f_1, f_2, Z, \epsilon_V, \epsilon_1, \epsilon_0\}.
\]

In either case, all of the information available to the agent is not required to satisfy conditional independence (M-1). All three information sets guarantee conditional independence, but only

\footnote{Notice that for a fixed set of \(\alpha_{ij}\), the minimal information set is \((\alpha_{11} - \alpha_{01}) f_1 + (\alpha_{12} - \alpha_{02}) f_2\), which captures the dependence between \(D\) and \((Y_1, Y_0)\).}
the first is minimal relevant.

In the notation of section 1, the observing economist may know some variables not in \(I_A, I_R\), or \(I_R\) but may not know all of the variables in \(I_R\). In the following subsections, I study what happens when the matching assumption that \(\sigma(I_E) \supseteq \sigma(I_R)\) does not hold. That is, I analyze what happens to the bias from matching as the amount of information used by the econometrician is changed. In order to get closed form expressions for the biases of the treatment parameters I make the additional assumption that

\[
(f_1, f_2, \varepsilon_V, \varepsilon_1, \varepsilon_0) \sim N(0, \Sigma),
\]

where \(\Sigma\) is a matrix with \((\sigma^2_{f_1}, \sigma^2_{f_2}, \sigma^2_{\varepsilon_V}, \sigma^2_{\varepsilon_1}, \sigma^2_{\varepsilon_0})\) on the diagonal and zero in all the non-diagonal elements. This assumption links matching models to conventional normal selection models. However, the examples based on this specification illustrate more general principles. I now analyze various commonly encountered cases.

### 3.5.1 The Economist Uses the Minimal Relevant Information: \(\sigma(I_R) \subseteq \sigma(I_E)\)

I begin by analyzing the case in which the information used by the economist is \(I_E = \{Z, f_1, f_2\}\), so that the econometrician has access to a relevant information set and it is larger than the minimal relevant information set. In this case, it is straightforward to show that matching identifies all of the mean treatment parameters with no bias. The matching estimator has population mean

\[
E(Y_1|D = 1, I_E) - E(Y_0|D = 0, I_E) = \mu_1 - \mu_0 + (\alpha_{11} - \alpha_{01}) f_1 + (\alpha_{12} - \alpha_{02}) f_2,
\]

and all of the mean treatment parameters collapse to this same expression since, conditional on knowing \(f_1\) and \(f_2\), there is no selection because \((\varepsilon_1, \varepsilon_0) \perp \perp V\). Recall that \(I_R = \{f_1, f_2\}\) and the economist needs less information to achieve (M-1) than is contained in \(I_E\) for arbitrary choices of \(\alpha_{11}, \alpha_{01}, \alpha_{12}, \alpha_{02}\).

In this case, the analysis of Rosenbaum and Rubin (1983) tells us that knowledge of \((Z, f_1, f_2)\) and knowledge of \(P(Z, f_1, f_2)\) are equally useful in identifying all of the treatment parameters.
conditional on \( P \). If we write the propensity score as

\[
P(I_E) = \Pr \left( \frac{\varepsilon_V}{\sigma_{ev}} > -Z \gamma - \alpha V_1 f_1 - \alpha V_2 f_2 \right) = 1 - \Phi \left( \frac{-Z \gamma - \alpha V_1 f_1 - \alpha V_2 f_2}{\sigma_{ev}} \right) = p,
\]

the event \( (D^* \preceq 0, \text{given } f = \tilde{f} \text{ and } Z = z) \) can be written as \( \frac{\varepsilon_V}{\sigma_{ev}} \preceq \Phi^{-1} (1 - P(z, \tilde{f})) \), where \( \Phi \) is the cdf of a standard normal random variable and \( f = (f_1, f_2) \). I abuse notation slightly by using \( z \) as the realized fixed value of \( Z \) and \( \tilde{f} \) as the realized value of \( f \). The population matching condition (M-1) implies that

\[
E(Y_1 | D = 1, P(I_E) = P(z, \tilde{f})) - E(Y_0 | D = 0, P(I_E) = P(z, \tilde{f})) = \mu_1 - \mu_0 + E \left( U_1 | \frac{\varepsilon_V}{\sigma_{ev}} > \Phi^{-1} (1 - P(z, \tilde{f})) \right) - E \left( U_0 | \frac{\varepsilon_V}{\sigma_{ev}} \leq \Phi^{-1} (1 - P(z, \tilde{f})) \right)
\]

This expression is equal to all of the treatment parameters discussed in this paper, since

\[
E \left( U_1 | \frac{\varepsilon_V}{\sigma_{ev}} > \Phi^{-1} (1 - P(z, \tilde{f})) \right) = \frac{\text{Cov}(U_1, \varepsilon_V)}{\sigma_{ev}} M_1 \left( P(z, \tilde{f}) \right)
\]

and

\[
E \left( U_0 | \frac{\varepsilon_V}{\sigma_{ev}} \leq \Phi^{-1} (1 - P(z, \tilde{f})) \right) = \frac{\text{Cov}(U_0, \varepsilon_V)}{\sigma_{ev}} M_0 \left( P(z, \tilde{f}) \right)
\]

where

\[
M_1(P(z, \tilde{f})) = \frac{\phi(\Phi^{-1}(1 - P(z, \tilde{f})))}{P(z, \tilde{f})}
\]

\[
M_0(P(z, \tilde{f})) = -\frac{\phi(\Phi^{-1}(1 - P(z, \tilde{f})))}{1 - P(z, \tilde{f})}
\]

where \( \phi \) is the density of a standard normal random variable. As a consequence of the assump-
tions about mutual independence of the errors

\[ \text{Cov} (U_i, \varepsilon_V) = \text{Cov} (\alpha_i f_1 + \alpha_2 f_2 + \varepsilon_i, \varepsilon_V) = 0, \quad i = 0, 1. \]

In the context of the generalized Roy model, the case considered in this subsection is the one matching is designed to solve. Even though a selection model generates the data, the fact that the information used by the econometrician includes the minimal relevant information makes matching a correct solution to the selection problem. One can estimate the treatment parameters with no bias since, as a consequence of the assumptions, \((U_1, U_0) \perp\perp D|(f, Z)\), which is exactly what matching requires. The minimal relevant information set is even smaller. For arbitrary factor loadings, one only needs to know \((f_1, f_2)\) to secure conditional independence. One can define the propensity score solely in terms of \(f_1\) and \(f_2\), and the Rosenbaum-Rubin result still goes through.

The analysis in this section focuses on treatment parameters conditional on particular values of \(P(Z, f) = P(z, \tilde{f})\), i.e., for fixed values of \(p\), but we could condition more finely. Conditioning on \(P(z, \tilde{f})\) defines the treatment parameters more coarsely. One can use either fine or coarse conditioning to construct the unconditional treatment effects.

In this example, using more information than what is in the relevant information set (i.e., using \(Z\)) is harmless. But this is not generally true. If \(Z \not\perp\perp (U_0, U_1, V)\), adding \(Z\) to the conditioning set can violate conditional independence assumption (M-1):

\[ (Y_0, Y_1) \perp\perp D \mid (f_1, f_2), \]

but

\[ (Y_0, Y_1) \not\perp\perp D \mid f_1, f_2, Z. \]

Adding extra variables can destroy the crucial conditional independence property of matching. I present an example of this point below. I first consider a case where \(Z \perp\perp (U_0, U_1, V)\) but the analyst conditions on \(Z\) and not \((f_1, f_2)\). In this case, there is selection on the unobservables that are not conditioned on.
3.5.2 The Economist does not Use All of the Minimal Relevant Information

Next, suppose that the information used by the econometrician is

\[ I_E = \{Z\}, \]

and there is selection on the unobservable (to the analyst) \( f_1 \) and \( f_2 \), i.e. the factor loadings \( \alpha_{ij} \) are all non zero. Recall that I assume that \( Z \) and the \( f \) are independent. In this case the event \((D^* \leq 0, Z = z)\) is characterized by

\[
\frac{\alpha_{V1} f_1 + \alpha_{V2} f_2 + \varepsilon_v}{\sqrt{\alpha_{V1}^2 \sigma^2_{f_1} + \alpha_{V2}^2 \sigma^2_{f_2} + \sigma^2_{\varepsilon_v}}} \leq \Phi^{-1} (1 - P(z)).
\]

Using the analysis presented in appendix A, the bias for the different treatment parameters is given by

\[
\text{Bias } TT(Z = z) = \text{Bias } TT(P(Z) = P(z)) = \eta_0 M(P(z)),
\]

where \( M(p) = M_1(P(z)) - M_0(P(z)) \).

\[
\text{Bias } ATE(Z = z) = \text{Bias } ATE(P(Z) = P(z)) = M(p(z))\{\eta_1 [1 - P(z)] + \eta_0 P(z)\},
\]

where

\[
\eta_1 = \frac{\alpha_{V1} \alpha_{11} \sigma^2_{f_1} + \alpha_{V2} \alpha_{12} \sigma^2_{f_2}}{\sqrt{\alpha_{V1}^2 \sigma^2_{f_1} + \alpha_{V2}^2 \sigma^2_{f_2} + \sigma^2_{\varepsilon_v}}},
\]

\[
\eta_0 = \frac{\alpha_{V1} \alpha_{01} \sigma^2_{f_1} + \alpha_{V2} \alpha_{02} \sigma^2_{f_2}}{\sqrt{\alpha_{V1}^2 \sigma^2_{f_1} + \alpha_{V2}^2 \sigma^2_{f_2} + \sigma^2_{\varepsilon_v}}}.\]

It is not surprising that matching on sets of variables that exclude the relevant conditioning variables produces bias for the conditional (on \( P(z) \)) treatment parameters. The advantage of working with a closed form expression for the bias is that it allows me to answer questions about the magnitude of this bias under different assumptions about the information available to the
analyst, and to present some simple examples. I next use expressions (23) and (24) as benchmarks against which to compare the relative size of the bias when I enlarge the econometrician’s information set beyond $Z$.

### 3.5.3 Adding Information to the Econometrician’s Information Set $I_E$: Using Some but not All the Information from the Minimal Relevant Information Set $I_R$

Suppose that the econometrician uses more information but not all of the information in the minimal relevant information set. He still reports values of the parameters conditional on specific $p$ values but now the model for $p$ has different conditioning variables. For example, the data set assumed in the preceding section might be augmented or else the econometrician decides to use information previously available. In particular, assume that the econometrician’s information set is

$$I'_E = \{Z, f_2\},$$

and that he uses this information set. Under conditions 1 and 2 presented below, the biases for the treatment parameters conditional on values of $P = p$ are reduced in absolute value relative to their values in section 3.5.2 by changing the conditioning set in this way. But these conditions are not generally satisfied, so that adding extra information does not necessarily reduce bias and may actually increase it. To show how this happens in the model considered here, I define expressions comparable to $\eta_1$ and $\eta_0$ for this case:

$$\eta'_1 = \frac{\alpha_{V1}\alpha_{11}\sigma^2_{f_1}}{\sqrt{\alpha_{V1}^2\sigma^2_{f_1} + \sigma_{eV}^2}},$$

$$\eta'_0 = \frac{\alpha_{V1}\alpha_{01}\sigma^2_{f_1}}{\sqrt{\alpha_{V1}^2\sigma^2_{f_1} + \sigma_{eV}^2}}.$$

I compare the biases under the two cases using formulas (23)–(24), suitably modified, but keeping $p$ fixed at a specific value even though this implies different conditioning sets in terms of $(z, \tilde{f})$.

**Condition 1** The bias produced by using matching to estimate TT is smaller in absolute value for any
given \( p \) when the new information set \( \sigma(I'_E) \) is used if

\[
|\eta_0| > |\eta'_0|.
\]

There is a similar result for ATE:

**Condition 2** The bias produced by using matching to estimate ATE is smaller in absolute value for any given \( p \) when the new information set \( \sigma(I'_E) \) is used if

\[
|\eta_1 (1 - p) + \eta_0 p| > |\eta'_1 (1 - p) + \eta'_0 p|.
\]

*Proof.* These conditions are a direct consequence of formulas (23) and (24), modified to allow for the different covariance structure produced by the information structure assumed in this section (replacing \( \eta_0 \) with \( \eta'_0 \), \( \eta_1 \) with \( \eta'_1 \)).

It is important to notice that I condition on the same value of \( p \) in deriving these expressions although the variables in \( P \) are different across different specifications of the model. Recall that propensity-score matching defines them conditional on \( P = p \).

These conditions do not always hold. In general, whether or not the bias will be reduced by adding additional conditioning variables depends on the relative importance of the additional information in both the outcome equations and on the signs of the terms inside the absolute value.

Consider whether Condition (1) is satisfied in general. Assume \( \eta_0 > 0 \) for all \( \alpha_{02}, \alpha_{v2} \). Then \( \eta_0 > \eta'_0 \) if

\[
\eta_0 = \frac{\alpha_{v1}\alpha_{01}\sigma^2_{f_1} + (\alpha_{v2})(\frac{\alpha_0}{\alpha_{v2}})\sigma^2_{f_2}}{\sqrt{\alpha_{v1}^2\sigma^2_{f_1} + \alpha_{v2}^2\sigma^2_{f_2} + \sigma^2_{\epsilon_v}}} > \frac{\alpha_{v1}\alpha_{11}\sigma^2_{f_1}}{\sqrt{\alpha_{v1}^2\sigma^2_{f_1} + \sigma^2_{\epsilon_v}}} = \eta'_0.
\]

When \( \frac{\alpha_0}{\alpha_{v2}} = 0 \), clearly \( \eta_0 < \eta'_0 \). Adding information to the conditioning set increases bias. I can vary \( \frac{\alpha_0}{\alpha_{v2}} \) holding all of the other parameters constant and hence can make the left hand side
arbitrarily large.\textsuperscript{56} As $\alpha_{02}$ increases, there is some critical value $\alpha^*_{02}$ beyond which $\eta_0 > \eta'_0$. If I assumed that $\eta_0 < 0$, however, the opposite conclusion would hold, and the conditions for reduction in bias would be harder to meet, as the relative importance of the new information is increased. Similar expressions can be derived for ATE and MTE, in which the direction of the effect depends on the signs of the terms in the absolute value.

Figures 9a and 9b illustrate the point that adding some but not all information from the minimal relevant set might increase the point-wise bias and the unconditional or average bias for ATE and TT, respectively.\textsuperscript{57} Values of the parameters of the model are presented at the base of the figures.

The fact that the point-wise (and overall) bias might increase when adding some but not all information from $I_R$ is a feature that is not shared by the method of control functions. Because the method of control functions models the stochastic dependence of the unobservables in the outcome equations on the observables, changing the variables observed by the econometrician to include $f_2$ does not generate bias. It only changes the control function used. That is, by adding $f_2$ one changes the control function from

\[
K_1(P(Z) = P(z)) = \eta_1 M_1(P(z)) \\
K_0(P(Z) = P(z)) = \eta_0 M_0(P(z))
\]

to

\[
K'_1(P(Z, f_2) = P(z, \tilde{f}_2)) = \eta'_1 M_1(P(z, \tilde{f}_2)) \\
K'_0(P(Z, f_2) = P(z, \tilde{f}_2)) = \eta'_0 M_0(P(z, \tilde{f}_2))
\]

but do not generate any bias by using the control function estimator. This is a major advantage

\textsuperscript{56}A direct computation shows that

\[
\frac{\partial \eta_0}{\partial \left( \frac{\alpha_{02}}{\alpha V_2} \right)} = \frac{\alpha_{v2}^2 \sigma_{\tilde{f}_2}^2}{\sqrt{\alpha_{v1}^2 \sigma_{f_1}^2 + \alpha_{v2}^2 \sigma_{f_2}^2 + \sigma_{v}^2}} > 0.
\]

\textsuperscript{57}Heckman and Navarro (2004) show comparable plots for MTE.
of this method. It controls for the bias of the omitted conditioning variables by modeling it. Of course, if the model for the bias term is not valid, neither is the correction for the bias. Semiparametric selection estimators are designed to protect the analyst against model misspecification. (See, e.g. Powell, 1994). Matching evades this problem by assuming that the analyst always knows the correct conditioning variables and that they satisfy (M-1). In actual empirical settings, agents rarely know the relevant information set. Instead they use proxies.

3.5.4 Adding Information to the Econometrician’s Information Set: Using Proxies for the Relevant Information

Suppose that instead of knowing some part of the minimal relevant information set, such as \( f_2 \), the analyst has access to a proxy for it.\(^{58}\) In particular, assume that he has access to a variable \( \tilde{Z} \) that is correlated with \( f_2 \) but that is not the full minimal relevant information set. That is, define the econometrician’s information to be

\[
\tilde{I}_E = \{ Z, \tilde{Z} \},
\]

and suppose that he uses it so \( I_E = \tilde{I}_E \). In order to obtain closed-form expressions for the biases I assume that

\[
\tilde{Z} \sim N(0, \sigma^2_{\tilde{Z}})
\]

\[
corr(\tilde{Z}, f_2) = \rho, \text{ and } \tilde{Z} \perp \perp (\varepsilon_0, \varepsilon_1, \varepsilon_V, f_1).
\]

Define expressions comparable to \( \eta \) and \( \eta' \):

\[
\tilde{\eta}_1 = \frac{\alpha_{11} \alpha_v \sigma_{f_1}^2 + \alpha_{12} \alpha_v \sigma_{f_2}^2 (1 - \rho^2)}{\sqrt{\alpha_{V1}^2 \sigma_{f_1}^2 + \alpha_{V2}^2 \sigma_{f_2}^2 (1 - \rho^2) + \sigma_{\varepsilon_V}^2}}
\]

\[
\tilde{\eta}_0 = \frac{\alpha_{01} \alpha_v \sigma_{f_1}^2 + \alpha_{02} \alpha_v \sigma_{f_2}^2 (1 - \rho^2)}{\sqrt{\alpha_{V1}^2 \sigma_{f_1}^2 + \alpha_{V2}^2 \sigma_{f_2}^2 (1 - \rho^2) + \sigma_{\varepsilon_V}^2}}
\]

\(^{58}\)For example, the returns-to-schooling literature often uses different test scores, like AFQT or IQ, to proxy for missing ability variables. These proxy, replacement function, methods are discussed in Heckman and Vytlacil (2007b, section 11) and in Abbring and Heckman (2007).
By substituting for $I_E'$ by $\tilde{I}_E$ and $\eta_j'$ by $\tilde{\eta}_j$ ($j = 0, 1$) in Conditions (1) and (2) of section 3.5.3 I can obtain results for the bias in this case. Whether $\tilde{I}_E$ will be bias-reducing depends on how well it spans $I_R$ and on the signs of the terms in the absolute values in those conditions in section 3.5.3.

In this case, however, there is another parameter to consider: the correlation $\rho$ between $\tilde{Z}$ and $f_2$. If $|\rho| = 1$ we are back to the case of $\tilde{I}_E = I_E'$ because $\tilde{Z}$ is a perfect proxy for $f_2$. If $\rho = 0$, we are essentially back to the case analyzed in section 3.5.3. Because we know that the bias at a particular value of $p$ might either increase or decrease when $f_2$ is used as a conditioning variable but $f_1$ is not, we know that it is not possible to determine whether the bias increases or decreases as we change the correlation between $f_2$ and $\tilde{Z}$. That is, we know that going from $\rho = 0$ to $|\rho| = 1$ might change the bias in any direction. Use of a better proxy in this correlational sense may produce a more biased estimate.

From the analysis of section 3.5.3, it is straightforward to derive conditions under which the bias generated when the econometrician’s information is $\tilde{I}_E$ is smaller than when it is $I_E'$. That is, it can be the case that knowing the proxy variable $\tilde{Z}$ is better than knowing the actual variable $f_2$. Take again the analysis of treatment on the treated as an example (i.e., Condition (1)). The bias in absolute value (at a fixed value of $p$) is reduced when $\tilde{Z}$ is used instead of $f_2$ if

$$\left| \frac{\alpha_{01} \alpha_{v1} \sigma_{f1}^2 + \alpha_{02} \alpha_{v2} (1 - \rho^2) \sigma_{f2}^2}{\sqrt{\alpha_{v1}^2 \sigma_{f1}^2 + \alpha_{v2}^2 \sigma_{f2}^2 (1 - \rho^2) + \sigma_{e}^2}} \right| < \left| \frac{\alpha_{01} \alpha_{v1} \sigma_{f1}^2}{\sqrt{\alpha_{v1}^2 \sigma_{f1}^2 + \sigma_{e}^2}} \right|.$$

Figures 10a and 10b, use the same true model as used in the previous section to illustrate the two points being made here. Namely, using a proxy for an unobserved relevant variable might increase the bias. On the other hand, it might be better in terms of bias to use a proxy than to use the actual variable, $f_2$. However, as figures 11a and 11b show, by changing $\alpha_{02}$ from 0.1 to 1, using a proxy might increase the bias versus using the actual variable $f_2$. Notice that the bias need not be universally negative or positive but depends on $p$.

The general point of these examples is that matching makes very knife-edge assumptions. If the analyst gets the right conditioning set, (M-1) is satisfied and there is no bias. But determining the correct information set is not a trivial task, as I note below. Having good proxies in the
standard usage of that term can create substantial bias in estimating treatment effects.

3.5.5 The Case of a Discrete Outcome Variable

The examples given in this section do not depend on all of the assumptions we have made to produce simple examples. In particular, we require neither normality nor additive separability of the outcome equations. The proposition that matching identifies the correct treatment if the econometrician’s information set includes all the minimal relevant information is true more generally, provided that any additional extraneous information used is exogenous in a sense to be defined precisely in the next section. Heckman and Navarro (2004) present parallel analyses of discrete treatment working with odds ratios and discrete outcomes that do not rely on either normality or separability of outcome equations.

3.5.6 On the Use of Model Selection Criteria to Choose Matching Variables

I have previously shown by way of example that adding more variables from the minimal relevant information set, but not all variables in it, may increase bias. By a parallel argument, adding additional variables to the relevant conditioning set may make the bias worse. Although I have used a prototypical Roy model as the point of departure, clearly the point is more general.

There is no rigorous rule for choosing the conditioning variables that produce (M-1). Adding variables that are statistically significant in the treatment choice equation is not guaranteed to select a set of conditioning variables that satisfies condition (M-1). This is demonstrated by the analysis of section 3.5.3 that shows that adding $f_2$ when it determines $D$ may increase bias at any selected value of $p$.

The existing literature (e.g. Heckman, Ichimura, Smith, and Todd, 1998) proposes criteria based on selecting a set of conditioning variables based on a goodness of fit criterion ($\lambda$), where a higher $\lambda$ means a better fit in the equation predicting $D$. The intuition behind such criteria is that by using some measure of goodness of fit as a guiding principle one is using information relevant to the decision process. In the example of section 3.5.3, it is clear that knowing $f_2$ improves goodness of fit, so that in general such a rule is deficient if $f_1$ is not known or is not
An implicit assumption underlying such procedures is that the added conditioning variables $X$ are exogenous in the following sense:

$$(Y_0, Y_1) \perp \perp D | I_{\text{int}}, X$$  \hspace{1cm} (E-1)

where $I_{\text{int}}$ is interpreted as the variables initially used as conditioning variables before $X$ is added. Failure of exogeneity is a failure of (M-1) for the augmented conditioning set, and matching estimators based on the augmented information set $(I_{\text{int}}, X)$ are biased when the condition is not satisfied.

Exogeneity assumption (E-1) is not usually invoked in the matching literature, which largely focuses on problem P-1, evaluating a program in place, rather than extrapolating to new environments (P-2). Indeed, the robustness of matching to such exogeneity assumptions is often viewed as an advantage of the method. In this section, I show some examples which illustrate the general point that standard model selection criteria fail to produce correctly specified conditioning sets unless some version of exogeneity condition (E-1) is satisfied.

In the literature, the use of model selection criteria is justified in two different ways. Sometimes it is claimed that they provide a relative guide. Sets of variables with higher $\lambda$ (better goodness of fit) are alleged to be better than sets of variables with lower $\lambda$ in the sense that they generate lower biases. However, I have already shown that this is not true. I know that enlarging the analyst’s information from $I_{\text{int}} = \{Z\}$ to $I'_{\text{int}} = \{Z, f_2\}$ will improve fit since $f_2$ is also in $I_A$ and $I_R$. But, going from $I_{\text{int}}$ to $I'_{\text{int}}$ might increase the bias. So it is not true that combinations of variables that increase some measure of fit $\lambda$ necessarily reduce the bias. Table 6 illustrates this point using the normal example. Going from row 1 to row 2 (adding $f_2$) improves goodness of fit and increases the unconditional or overall bias for all three treatment parameters, because (E-1) is violated.

The following rule of thumb argument is sometimes invoked as an absolute standard against which to compare alternative models. In versions of the argument, the analyst asserts that there is a combination of variables $I''$ that satisfy (M-1) and hence produces zero bias and a value of
\( \lambda = \lambda'' \) larger than that of any other \( I \). In the examples, conditioning on \( \{Z, f_1, f_2\} \) generates zero bias. One can exclude \( Z \) and still obtain zero bias. Because \( Z \) is a determinant of \( D \), this shows immediately that the best fitting model does not necessarily identify the minimal relevant information set. In this example including \( Z \) is innocuous because there is still zero bias and the added conditioning variables satisfies (E-1) where \( I_{\text{int}} = (f_1, f_2) \). In general, such a rule is not innocuous if \( Z \) is not exogenous. If goodness of fit is used as a rule to choose variables on which to match, there is no guarantee it produces a desirable conditioning set. If one includes in the conditioning set variables \( X \) that violate (E-1), they may improve the fit of predicted probabilities but worsen the bias.

Heckman and Navarro (2004) produce a series of examples that have the following feature. Variables \( S \) (shown at the base of table 6) are added to the information set that improve the prediction of \( D \) but are correlated with \( (U_0, U_1) \). Their particular examples use imperfect proxies \( (S_1, S_2) \) for \( (f_1, f_2) \). The point is more general. The \( S \) variables fail exogeneity and produce greater bias for TT and ATE but they improve the prediction of \( D \) as measured by the correct in-sample prediction rate and the pseudo-\( R^2 \). See the bottom two rows of table 6.

4 Summary and Conclusions

This paper exposits the key identifying assumptions of commonly used econometric evaluation estimators. The emphasis is on the economic content of these assumptions. Using the prototypical generalized Roy model as the point of departure, I examine the assumptions underlying the method of matching. In brief, they are: (a) the analyst knows the relevant information set, (b) a randomization can be produced by conditioning on regressors, and (c) conditioning on observables, marginal returns are the same as average returns. Other econometric estimators do not impose (c) and account for failures of (a) and (b), and in this sense are more general.
A Selection Formula for the Matching Examples

Consider a generalized Roy model of the form $Y_1 = \mu_1 + U_1; Y_0 = \mu_0 + U_0; D^* = \mu_D(Z) + V; D = 1$ if $D^* \geq 0$, $= 0$ otherwise; and $Y = DY_1 + (1 - D)Y_0$, where

\[
(U_1, U_0, V)' \sim N(0, \Sigma); \text{Var}(U_i) = \sigma_i^2 \quad i = 0, 1 \\
\text{Var}(V) = \sigma_V^2; \text{Cov}(U_1, U_0) = \sigma_{10} \\
\text{Cov}(U_1, V) = \sigma_{1V}; \text{Cov}(U_0, V) = \sigma_{0V}.
\]

Assume $Z \perp \perp (U_0, U_1, V)$. Let $\phi(\cdot)$ and $\Phi(\cdot)$ be the pdf and the cdf of a standard normal random variable. Then, the propensity score for this model for $Z = z$ is given by:

\[
\Pr(D^* > 0 | Z = z) = \Pr(V > -\mu_D(z)) = P(z) = \Phi\left(\frac{\mu_D(z)}{\sigma_V}\right).
\]

Thus $\frac{\mu_D(z)}{\sigma_V} = \Phi^{-1}(P(z))$, and

\[
-\frac{\mu_D(z)}{\sigma_V} = \Phi^{-1}(1 - P(z)).
\]

The event $(V \leq 0, Z = z)$ can be written as $\frac{V}{\sigma_V} \leq -\frac{\mu_D(z)}{\sigma_V} \iff \frac{V}{\sigma_V} \leq \Phi^{-1}(1 - P(z))$. We can write the conditional expectations required to get the biases for the treatment parameters as a function of $P(z) = p$. For $U_1$:

\[
E(U_1 | D^* > 0, Z = z) = \frac{\sigma_{1V}}{\sigma_V} E\left(\frac{V}{\sigma_V} \bigg| \frac{V}{\sigma_V} > -\frac{\mu_D(z)}{\sigma_V}\right) \\
= \frac{\sigma_{1V}}{\sigma_V} E\left(\frac{V}{\sigma_V} \bigg| \frac{V}{\sigma_V} > \Phi^{-1}(1 - P(z))\right) \\
= \eta_1 M_1(P(z))
\]

where

\[
\eta_1 = \frac{\sigma_{1V}}{\sigma_V}.
\]
Similarly for $U_0$:

$$E(U_0|D^* > 0, Z = z) = \eta_0 M_1(P(z))$$
$$E(U_0|D^* < 0, Z = z) = \eta_0 M_0(P(z)),$$

where $\eta_0 = \frac{\sigma_{uv}}{\sigma_v}$ and $M_1(P(z)) = \frac{\phi(\Phi^{-1}(1-P(z)))}{P(z)}$ and $M_0(P(z)) = -\frac{\phi(\Phi^{-1}(1-P(z)))}{1-P(z)}$ are inverse Mills ratio terms.

Substituting these into the expressions for the biases for the treatment parameters conditional on $z$ we obtain

$$\text{Bias } TT (P(z)) = \eta_0 M_1(P(z)) - \eta_0 M_0(P(z))$$
$$= \eta_0 M(P(z)),$$

$$\text{Bias } ATE (P(z)) = \eta_1 M_1(P(z)) - \eta_0 M_0(P(z))$$
$$= M(P(z)) (\eta_1 (1 - P(z)) + \eta_0 P(z)).$$
References


List of tables

1. Identifying assumptions under commonly used methods.
2. A. Treatment effects and estimands as weighted averages of the marginal treatment effect; B. Weights. Source: Heckman and Vytlacil (2005)

List of figures

2. A. Weights for the marginal treatment effect for different parameters; B. Marginal treatment effect vs. linear instrumental variables and ordinary least squares weights. Source: Heckman and Vytlacil (2005)
3. Conditional expectation of \( Y \) on \( P(Z) \) and the MTE, the extended Roy economy. Source: Heckman, Urzua, and Vytlacil (2006)
4. A. Plot of the \( E(Y \mid P(Z) = p) \); B. Plot of the identified marginal treatment effect from figure 4A (the derivative). Source: Heckman and Vytlacil (2005)
5. The local average treatment effect, the extended Roy economy. Source: Heckman, Urzua, and Vytlacil (2006)
7. The least squares extrapolation problem avoided by using nonparametric regression or matching
8. A. MTE vs. linear instrumental variables, ordinary least squares, and policy relevant treatment effect weights: when \( P(Z) \) is the instrument; B. MTE vs. linear IV with \( P(Z(1 + t(1[Z > 0]))) = P(Z, t) \) as an instrument, and policy relevant treatment effect weights; C. MTE vs. IV policy and policy relevant treatment effect weights. Source: Heckman and Vytlacil (2005)
Table 1
Identifying Assumptions Under Commonly Used Methods

\( (Y_0, Y_1) \) are potential outcomes that depend on \( X \)

\[
D = \begin{cases} 
1 & \text{if assigned (or choose) status 1} \\
0 & \text{otherwise}
\end{cases}
\]

\( Z \) are determinants of \( D \), \( \theta \) is a vector of unobservables

For random assignments, \( A \) is a vector of actual treatment status. \( A = 1 \) if treated; \( A = 0 \) if not.

\( \xi = 1 \) if a person is randomized to treatment status; \( \xi = 0 \) otherwise.

<table>
<thead>
<tr>
<th>Identifying Assumptions</th>
<th>Identifies marginal distributions?</th>
<th>Exclusion condition needed?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Random Assignment ((Y_0, Y_1) \perp \xi, \xi = 1 \implies A = 1, \xi = 0 \implies A = 0) (full compliance)</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td></td>
<td>Alternatively, if self-selection is random with respect to outcomes, ((Y_0, Y_1) \perp D,) Assignment can be conditional on ( X ).</td>
<td></td>
</tr>
<tr>
<td>Matching ((Y_0, Y_1) \not\perp D, \text{ but } (Y_0, Y_1) \perp D \mid X, ) ( 0 &lt; \Pr(D = 1 \mid X) &lt; 1 ) for all ( X ) So ( D ) conditional on ( X ) is a nondegenerate random variable</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>Control Functions and Extensions ((Y_0, Y_1) \not\perp D \mid X, Z, \text{ but } (Y_1, Y_0) \perp D \mid X, Z, \theta. ) The method models dependence induced by ( \theta ) or else proxies ( \theta ) (replacement function) Version (i) Replacement functions (substitute out ( \theta ) by observables) (Blundell and Powell, 2003; Heckman and Robb, 1985; Olley and Pakes, 1994). Factor models (Carneiro, Hansen and Heckman, 2003) allow for measurement error in the proxies. Version (ii) Integrate out ( \theta ) assuming ( \theta \perp \perp (X, Z) ) (Aakvik, Heckman, and Vytlacil, 2005; Carneiro, Hansen, and Heckman, 2003) Version (iii) For separable models for mean response expect ( \theta ) conditional on ( X, Z, D ) as in standard selection models (control functions in the same sense of Heckman and Robb).</td>
<td>Yes</td>
<td>Yes (for semiparametric models)</td>
</tr>
<tr>
<td>IV ((Y_0, Y_1) \not\perp D \mid X, Z, \text{ but } (Y_1, Y_0) \perp D \mid X, ) ( \Pr(D = 1 \mid Z) ) is a nondegenerate function of ( Z )</td>
<td>Yes</td>
<td>Yes</td>
</tr>
</tbody>
</table>

Source: Heckman and Vytlacil (2007b)
Table 2A
Treatment Effects and Estimands as Weighted Averages of the Marginal Treatment Effect

\[
\text{ATE}(x) = E(Y_1 - Y_0 \mid X = x) = \int_0^1 \Delta_{\text{MTE}}(x, u_D) \, du_D
\]

\[
\text{TT}(x) = E(Y_1 - Y_0 \mid X = x, D = 1) = \int_0^1 \Delta_{\text{MTE}}(x, u_D) \omega_{\text{TT}}(x, u_D) \, du_D
\]

\[
\text{TUT}(x) = E(Y_1 - Y_0 \mid X = x, D = 0) = \int_0^1 \Delta_{\text{MTE}}(x, u_D) \omega_{\text{TUT}}(x, u_D) \, du_D
\]

Policy Relevant Treatment Effect \( (x) = E(Y_{a'} \mid X = x) - E(Y_a \mid X = x) = \int_0^1 \Delta_{\text{MTE}}(x, u_D) \omega_{\text{PRTE}}(x, u_D) \, du_D \)

for two policies \( a \) and \( a' \) that affect the \( Z \) but not the \( X \)

\[
\text{IV}_J(x) = \int_0^1 \Delta_{\text{MTE}}(x, u_D) \omega_{\text{IV}}(x, u_D) \, du_D, \text{ given instrument } J
\]

\[
\text{OLS}(x) = \int_0^1 \Delta_{\text{MTE}}(x, u_D) \omega_{\text{OLS}}(x, u_D) \, du_D
\]

Table 2B
Weights

\[
\omega_{\text{ATE}}(x, u_D) = 1
\]

\[
\omega_{\text{TT}}(x, u_D) = \left[ \int_{u_D}^1 f(p \mid X = x) \, dp \right] \frac{1}{E(P \mid X = x)}
\]

\[
\omega_{\text{TUT}}(x, u_D) = \left[ \int_{u_D}^0 f(p \mid X = x) \, dp \right] \frac{1}{E((1 - P) \mid X = x)}
\]

\[
\omega_{\text{PRTE}}(x, u_D) = \left[ \frac{F_{P_{a'},X}(u_D) - F_{P_a,X}(u_D)}{\Delta P} \right]
\]

\[
\omega_{\text{IV}}(x, u_D) = \left[ \int_{u_D}^1 J(Z) - E(J(Z) \mid X = x) \right] \frac{1}{\text{Cov}(J(Z), D \mid X = x)}
\]

\[
\omega_{\text{OLS}}(x, u_D) = 1 + \frac{E(U_1 \mid X = x, U_D = u_D) \omega_1(x, u_D) - E(U_0 \mid X = x, U_D = u_D) \omega_0(x, u_D)}{\Delta_{\text{MTE}}(x, u_D)}
\]

\[
\omega_1(x, u_D) = \left[ \int_{u_D}^1 f(p \mid X = x) \, dp \right] \frac{1}{E(P \mid X = x)}
\]

\[
\omega_0(x, u_D) = \left[ \int_{u_D}^0 f(p \mid X = x) \, dp \right] \frac{1}{E((1 - P) \mid X = x)}
\]

Source: Heckman and Vytlacil (2005)
Table 3
Treatment Parameters and Estimands in the Generalized Roy Example

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Treatment on the Treated</td>
<td>0.2353</td>
</tr>
<tr>
<td>Treatment on the Untreated</td>
<td>0.1574</td>
</tr>
<tr>
<td>Average Treatment Effect</td>
<td>0.2000</td>
</tr>
<tr>
<td>Sorting Gain(^a)</td>
<td>0.0353</td>
</tr>
<tr>
<td>Policy Relevant Treatment Effect (PRTE)</td>
<td>0.1549</td>
</tr>
<tr>
<td>Selection Bias(^b)</td>
<td>−0.0628</td>
</tr>
<tr>
<td>Linear Instrumental Variables(^c)</td>
<td>0.2013</td>
</tr>
<tr>
<td>Ordinary Least Squares</td>
<td>0.1725</td>
</tr>
</tbody>
</table>

\(^a\) \(TT - ATE = E(Y_1 - Y_0 \mid D = 1) - E(Y_1 - Y_0)\)
\(^b\) \(OLS - TT = E(Y_0 \mid D = 1) - E(Y_0 \mid D = 0)\)
\(^c\) Using Propensity Score \(P(Z)\) as the instrument.

Note: The model used to create Table 2 is the same as those used to create Figures 1A and 1B. The PRTE is computed using a policy \(t\) characterized as follows:
If \(Z > 0\) then \(D = 1\) if \(Z(1 + t) - V > 0\).
If \(Z \leq t\) then \(D = 1\) if \(Z - V > 0\).
For this example \(t\) is set equal to 0.2.

Source: Heckman and Vytlacil (2005)
Table 4
Mean Bias for Treatment on the Treated

<table>
<thead>
<tr>
<th>$\rho_0 V$</th>
<th>Average Bias ($\sigma_0 = 1$)</th>
<th>Average Bias ($\sigma_0 = 2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.00</td>
<td>-1.7920</td>
<td>-3.5839</td>
</tr>
<tr>
<td>-0.75</td>
<td>-1.3440</td>
<td>-2.6879</td>
</tr>
<tr>
<td>-0.50</td>
<td>-0.8960</td>
<td>-1.7920</td>
</tr>
<tr>
<td>-0.25</td>
<td>-0.4480</td>
<td>-0.8960</td>
</tr>
<tr>
<td>0.00</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.25</td>
<td>0.4480</td>
<td>0.8960</td>
</tr>
<tr>
<td>0.50</td>
<td>0.8960</td>
<td>1.7920</td>
</tr>
<tr>
<td>0.75</td>
<td>1.3440</td>
<td>2.6879</td>
</tr>
<tr>
<td>1.00</td>
<td>1.7920</td>
<td>3.5839</td>
</tr>
</tbody>
</table>

$\text{BIAS\ TT} = \rho_0 V \ast \sigma_0 \ast M(p)$

$M(p) = \frac{\phi(\Phi^{-1}(p))}{\sqrt{p(1-p)}}$

Table 5
Mean Bias for Average Treatment Effect
\((\sigma_0 = 1)\)

<table>
<thead>
<tr>
<th>(\rho_0V)</th>
<th>-1.00</th>
<th>-0.75</th>
<th>-0.50</th>
<th>-0.25</th>
<th>0</th>
<th>0.25</th>
<th>0.50</th>
<th>0.75</th>
<th>1.00</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\rho_1V(\sigma_1 = 1))</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-1.00</td>
<td>-1.7920</td>
<td>-1.5680</td>
<td>-1.3440</td>
<td>-1.1200</td>
<td>-0.8960</td>
<td>-0.6720</td>
<td>-0.4480</td>
<td>-0.2240</td>
<td>0</td>
</tr>
<tr>
<td>-0.75</td>
<td>-1.5680</td>
<td>-1.3440</td>
<td>-1.1200</td>
<td>-0.8960</td>
<td>-0.6720</td>
<td>-0.4480</td>
<td>-0.2240</td>
<td>0</td>
<td>0.2240</td>
</tr>
<tr>
<td>-0.50</td>
<td>-1.3440</td>
<td>-1.1200</td>
<td>-0.8960</td>
<td>-0.6720</td>
<td>-0.4480</td>
<td>-0.2240</td>
<td>0</td>
<td>0.2240</td>
<td>0.4480</td>
</tr>
<tr>
<td>-0.25</td>
<td>-1.1200</td>
<td>-0.8960</td>
<td>-0.6720</td>
<td>-0.4480</td>
<td>-0.2240</td>
<td>0</td>
<td>0.2240</td>
<td>0.4480</td>
<td>0.6720</td>
</tr>
<tr>
<td>0</td>
<td>-0.8960</td>
<td>-0.6720</td>
<td>-0.4480</td>
<td>-0.2240</td>
<td>0</td>
<td>0.2240</td>
<td>0.4480</td>
<td>0.6720</td>
<td>0.8960</td>
</tr>
<tr>
<td>0.25</td>
<td>-0.6720</td>
<td>-0.4480</td>
<td>-0.2240</td>
<td>0</td>
<td>0.2240</td>
<td>0.4480</td>
<td>0.6720</td>
<td>0.8960</td>
<td>1.1200</td>
</tr>
<tr>
<td>0.50</td>
<td>-0.4480</td>
<td>-0.2240</td>
<td>0</td>
<td>0.2240</td>
<td>0.4480</td>
<td>0.6720</td>
<td>0.8960</td>
<td>1.1200</td>
<td>1.3440</td>
</tr>
<tr>
<td>0.75</td>
<td>-0.2240</td>
<td>0</td>
<td>0.2240</td>
<td>0.4480</td>
<td>0.6720</td>
<td>0.8960</td>
<td>1.1200</td>
<td>1.3440</td>
<td>1.5680</td>
</tr>
<tr>
<td>1.00</td>
<td>0</td>
<td>0.2240</td>
<td>0.4480</td>
<td>0.6720</td>
<td>0.8960</td>
<td>1.1200</td>
<td>1.3440</td>
<td>1.5680</td>
<td>1.7920</td>
</tr>
</tbody>
</table>

| \(\rho_1V(\sigma_1 = 2)\) |      |      |      |      |     |      |      |      |      |
| -1.00    | -2.6879 | -2.2399 | -1.7920 | -1.3440 | -0.8960 | -0.4480 | 0     | 0.4480 | 0.8960 |
| -0.75    | -2.4639 | -2.0159 | -1.5680 | -1.1200 | -0.6720 | -0.2240 | 0.2240 | 0.6720 | 1.1200 |
| -0.50    | -2.2399 | -1.7920 | -1.3440 | -0.8960 | -0.4480 | 0     | 0.4480 | 0.8960 | 1.3440 |
| -0.25    | -2.0159 | -1.5680 | -1.1200 | -0.6720 | -0.2240 | 0.2240 | 0.6720 | 1.1200 | 1.5680 |
| 0        | -1.7920 | -1.3440 | -0.8960 | -0.4480 | 0     | 0.4480 | 0.8960 | 1.3440 | 1.7920 |
| 0.25     | -1.5680 | -1.1200 | -0.6720 | -0.2240 | 0.2240 | 0.6720 | 1.1200 | 1.5680 | 2.0159 |
| 0.50     | -1.3440 | -0.8960 | -0.4480 | 0     | 0.4480 | 0.8960 | 1.3440 | 1.7920 | 2.2399 |
| 0.75     | -1.1200 | -0.6720 | -0.2240 | 0.2240 | 0.6720 | 1.1200 | 1.5680 | 2.0159 | 2.4639 |
| 1.00     | -0.8960 | -0.4480 | 0     | 0.4480 | 0.8960 | 1.3440 | 1.7920 | 2.2399 | 2.6879 |

**BIAS ATE** = \(\rho_1V \ast \sigma_1 \ast M_1(p) - \rho_0V \ast \sigma_0 \ast M_0(p)\)

\[M_1(p) = \frac{\phi(\Phi^{-1}(p))}{p}\]

\[M_0(p) = \frac{-\phi(\Phi^{-1}(p))}{1-p}\]

Table 6

<table>
<thead>
<tr>
<th>Variables in Probit</th>
<th>Goodness of fit statistics $\lambda$</th>
<th>Average Bias</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Correct in-sample prediction rate</td>
<td>Pseudo $R^2$</td>
</tr>
<tr>
<td>$Z$</td>
<td>66.88%</td>
<td>0.1284</td>
</tr>
<tr>
<td>$Z, f_2$</td>
<td>75.02%</td>
<td>0.2791</td>
</tr>
<tr>
<td>$Z, f_1, f_2$</td>
<td>83.45%</td>
<td>0.4844</td>
</tr>
<tr>
<td>$Z, S_1$</td>
<td>77.38%</td>
<td>0.3282</td>
</tr>
<tr>
<td>$Z, S_2$</td>
<td>92.25%</td>
<td>0.7498</td>
</tr>
</tbody>
</table>

Figure 1. Distribution of Gains

The Roy Economy

$U_1 - U_0 \not\sim D$

TT = 2.666, TUT = −0.632

Return to Marginal Agent = $C = 1.5$

ATE = $\mu_1 - \mu_0 = \bar{\beta} = 0.2$

The Model

<table>
<thead>
<tr>
<th>Outcomes</th>
<th>Choice Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y_1 = \mu_1 + U_1 = \alpha + \bar{\beta} + U_1$</td>
<td>$D = \begin{cases} 1 &amp; \text{if } D^* &gt; 0 \ 0 &amp; \text{if } D^* \leq 0 \end{cases}$</td>
</tr>
<tr>
<td>$Y_0 = \mu_0 + U_0 = \alpha + U_0$</td>
<td></td>
</tr>
</tbody>
</table>

General Case

$(U_1 - U_0) \not\sim D$

ATE $\neq$ TT $\neq$ TUT

The Researcher Observes $(Y, D, C)$

$Y = \alpha + \beta D + U_0$ where $\beta = Y_1 - Y_0$

Parameterization

$\alpha = 0.67$  $(U_1, U_0) \sim N(0, \Sigma)$  $D^* = Y_1 - Y_0 - C$

$\bar{\beta} = 0.2$  $\Sigma = \begin{bmatrix} 1 & -0.9 \\ -0.9 & 1 \end{bmatrix}$  $C = 1.5$

Source: Heckman, Urzua and Vytlacil (2006)
**Figure 2A**  
Weights for the Marginal Treatment Effect for Different Parameters

**Figure 2B**  
Marginal Treatment Effect vs Linear Instrumental Variables and Ordinary Least Squares Weights

\[
\begin{align*}
Y_1 &= \alpha + \beta + U_1 \\
Y_0 &= \alpha + U_0 \\
D &= 1 \text{ if } Z - V > 0 \\
U_1 &= \sigma_1 \varepsilon \\
U_0 &= \sigma_0 \varepsilon \\
V &= \sigma_V \varepsilon \\
\varepsilon &\sim N(0, 1) \\
U_D &= \Phi \left( \frac{V}{\sigma_V \sigma_\varepsilon} \right) \\
Z &\sim N(-0.0026, 0.2700)
\end{align*}
\]

Source: Heckman and Vytlacil (2005)
Figure 3. Conditional Expectation of $Y$ on $P(Z)$ and the Marginal Treatment Effect (MTE) 
The Extended Roy Economy

A. $E(Y|P(Z) = p)$  

B. $\Delta^{MTE}(u_D)$

Outcomes | Choice Model  
---|---  
$Y_1 = \alpha + \bar{\beta} + U_1$ | $D = \begin{cases} 
1 & \text{if } D^* > 0 \\
0 & \text{if } D^* \leq 0 
\end{cases}$  
$Y_0 = \alpha + U_0$ 

<table>
<thead>
<tr>
<th>Case IA</th>
<th>Case IB</th>
<th>Case II</th>
</tr>
</thead>
</table>
$U_1 = U_0$ | $U_1 - U_0 \perp D$ | $U_1 - U_0 \not\perp D$ |
$\bar{\beta} = \text{ATE=TT=TUT=IV}$ | $\bar{\beta} = \text{ATE=TT=TUT=IV}$ | $\bar{\beta} = \text{ATE\not=TT\not=TUT\not=IV}$ |

Parameterization

<table>
<thead>
<tr>
<th>Cases IA, IB and II</th>
<th>Cases IB and II</th>
<th>Case II</th>
</tr>
</thead>
</table>
$\alpha = 0.67$ | $(U_1, U_0) \sim N(0, \Sigma)$ | $D^* = Y_1 - Y_0 - \gamma Z$ |
$\bar{\beta} = 0.2$ | with $\Sigma = \begin{bmatrix} 1 & -0.9 \\ -0.9 & 1 \end{bmatrix}$ | $Z \sim N(\mu_Z, \Sigma_Z)$ |
$\mu_Z = (2, -2)$ and $\Sigma_Z = \begin{bmatrix} 9 & -2 \\ -2 & 9 \end{bmatrix}$ | $\gamma = (0.5, 0.5)$ |

Source: Heckman, Urzua and Vytlacil (2006)
Figure 4A
Plot of the $E(Y|P(Z) = p)$

Figure 4B
Plot of the Identified Marginal Treatment Effect from Figure 2A (the Derivative).

Note: Parameters for the general heterogeneous case are the same as those used in Figures 2A and 2B. For the homogeneous case we impose $U_1 = U_0$ ($\sigma_1 = \sigma_0 = 0.012$).

Source: Heckman and Vytlacil (2005)
Figure 5. The Local Average Treatment Effect
The Extended Roy Economy

A. $E(Y|P(Z) = p)$ and $\Delta^{LATE}(p_\ell, p_{\ell+1})$

B. $\Delta^{MTE}(u_D)$ and $\Delta^{LATE}(p_\ell, p_{\ell+1})$

$\Delta^{LATE}(p_\ell, p_{\ell+1}) = \frac{E(Y|P(Z) = p_{\ell+1}) - E(Y|P(Z) = p_\ell)}{p_{\ell+1} - p_\ell}$

$\int_{p_\ell}^{p_{\ell+1}} \frac{\Delta^{MTE}(u_D)}{p_{\ell+1} - p_\ell} du_D$

$\Delta^{LATE}(0.6, 0.9) = -1.17$
$\Delta^{LATE}(0.1, 0.35) = 1.719$

Outcomes

$Y_1 = \alpha + \beta + U_1$
$Y_0 = \alpha + U_0$

Choice Model

$D = \begin{cases} 
1 & \text{if } D^* > 0 \\
0 & \text{if } D^* \leq 0 
\end{cases}$

with $D^* = Y_1 - Y_0 - \gamma Z$

Parameterization

$(U_1, U_0) \sim N(0, \Sigma)$ and $Z \sim N(\mu_Z, \Sigma_Z)$

$\Sigma = \begin{bmatrix} 
1 & -0.9 \\
-0.9 & 1 
\end{bmatrix}$, $\mu_Z = (2, -2)$ and $\Sigma_Z = \begin{bmatrix} 
9 & -2 \\
-2 & 9 
\end{bmatrix}$

$\alpha = 0.67, \beta = 0.2, \gamma = (0.5, 0.5)$

Source: Heckman, Urzua and Vytlacil (2006)
Figure 6. Treatment Parameters and OLS/Matching as a function of $P(Z) = p$

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Definition</th>
<th>Under Assumptions (*)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Marginal Treatment Effect</td>
<td>$E[Y_1 - Y_0</td>
<td>D^* = 0, P(Z) = p]$</td>
</tr>
<tr>
<td>Average Treatment Effect</td>
<td>$E[Y_1 - Y_0</td>
<td>P(Z) = p]$</td>
</tr>
<tr>
<td>Treatment on the Treated</td>
<td>$E[Y_1 - Y_0</td>
<td>D^* &gt; 0, P(Z) = p]$</td>
</tr>
<tr>
<td>Treatment on the Untreated</td>
<td>$E[Y_1 - Y_0</td>
<td>D^* \leq 0, P(Z) = p]$</td>
</tr>
<tr>
<td>OLS/Matching on $P(Z)$</td>
<td>$E[Y_1</td>
<td>D^* &gt; 0, P(Z) = p] - E[Y_0</td>
</tr>
</tbody>
</table>

Note: $\Phi(\cdot)$ and $\phi(\cdot)$ represent the cdf and pdf of a standard normal distribution, respectively. $\Phi^{-1}(\cdot)$ represents the inverse of $\Phi(\cdot)$.

(*): The model in this case is the same as the one presented below Figure 5.

Source: Heckman, Urzua and Vytlacil (2006)
Figure 7. The Least Squares Extrapolation Problem Avoided by Using Nonparametric Regression or Matching

Source: Heckman and Vytlacil (2007b)
Figure 8A
Marginal Treatment Effect vs Linear Instrumental Variables, Ordinary Least Squares, and Policy Relevant Treatment Effect Weights: When $P(Z)$ is the Instrument
The Policy is Given at the Base of Table 3
The model parameters are given at the base of Figure 2

Source: Heckman and Vytlacil (2005)
Note: Using proxy $\hat{Z}$ for $f_2$ increases the bias. Correlation $(\hat{Z}, f_2) = 0.5$.

Model:

\[
V = Z + f_1 + f_2 + \varepsilon_\nu; \quad Y_1 = 2f_1 + 0.1f_2 + \varepsilon_1; \quad Y_0 = f_1 + 0.1f_2 + \varepsilon_0
\]

$\varepsilon_\nu \sim N(0, 1)$; $\varepsilon_1 \sim N(0, 1)$; $\varepsilon_0 \sim N(0, 1)$

$f_1 \sim N(0, 1)$; $f_2 \sim N(0, 1)$

Note: Using proxy $\tilde{Z}$ for $f_2$ increases the bias. Correlation $(\tilde{Z}, f_2) = 0.5$.

Model:

$$V = Z + f_1 + f_2 + \varepsilon_\nu; \quad Y_1 = 2f_1 + 0.1f_2 + \varepsilon_1; \quad Y_0 = f_1 + 0.1f_2 + \varepsilon_0$$

$$\varepsilon_\nu \sim N(0, 1); \quad \varepsilon_1 \sim N(0, 1); \quad \varepsilon_0 \sim N(0, 1)$$

$$f_1 \sim N(0, 1); \quad f_2 \sim N(0, 1)$$

Figure 11A. Bias for Treatment on the Treated

Figure 11B. Bias for Average Treatment Effect

Note: Using proxy $\tilde{Z}$ for $f_2$ increases the bias. Correlation ($\tilde{Z}, f_2$) = 0.5.

Model:

$V = Z + f_1 + f_2 + \varepsilon_\nu$; \hspace{1em} $Y_1 = 2f_1 + 0.1f_2 + \varepsilon_1$; \hspace{1em} $Y_0 = f_1 + 0.1f_2 + \varepsilon_0$

$\varepsilon_\nu \sim N(0, 1)$; \hspace{1em} $\varepsilon_1 \sim N(0, 1)$; \hspace{1em} $\varepsilon_0 \sim N(0, 1)$

$f_1 \sim N(0, 1)$; \hspace{1em} $f_2 \sim N(0, 1)$