

# Supplement to “Understanding Instrumental Variables in Models with Essential Heterogeneity”\*

James J. Heckman  
University of Chicago  
and American Bar Foundation

Sergio Urzua  
University of Chicago

Edward Vytlacil  
Columbia University

March 20, 2006

## I The Construction of Counterfactual Probabilities

Definition of the counterfactual  $D_j$ . This all takes place at the level of hypothetical manipulations. The space may have a probability measure or just a measure. For some  $J(Z) = j$ , define  $A_j(z) = \{z \mid J(z) = j\}$ . For each element in the set, we “project”  $\mu_D(z)$  onto the set. If we embed the set in  $L^2$ , this is clear. Otherwise, there may be other constructions (*e.g.*, embed in space  $L^P$  and for each norm take a projection. Take the infimum of these norms.) Such projections do not necessarily define a unique  $D_j$ . Consider  $z = (z_1, z_2)$ ,  $\mu_D(z) = \gamma z$  and  $J(z_1) = z_1$ . Then  $D_j$  is undefined. To make a unique construction we need to supplement the model of hypothetical counterfactuals. One supplement is a “rational expectations” type supplement. We fix  $J(z) = j$  and draw a  $z$  at random from this hypothetical distribution, conditional on the realized  $J(z) = j$ . This entails constructing a distribution for  $z$ . One such construction is the actual distribution of  $z_2$ , given  $z_1$  in the true population. With this  $z$  construction

$$D_j = \mathbf{1}(\mu_D(z) > U_D).$$

Now  $z$  is a random variable from the distribution given by picking  $J(z)$  and then drawing another element. Under (A-2),  $J(Z) \perp\!\!\!\perp (U_0, U_1, U_D)$ . The counterfactual manipulation in this case corresponds to picking (fixing) a  $J(Z) = j$ , drawing a  $Z$  at random from the population distribution. Insert the chosen  $Z$  into  $\mu_D(z)$ , and for a fixed  $U_D$  compute  $D_j$ .

Notice that there are other ways to define the  $D_j$ . We can draw from another distribution not necessarily corresponding to the population distribution for  $Z$ , or we can use deterministic sampling rules. If we do so, the probability from the hypothetical distribution will not necessarily correspond with the observed probability (*i.e.* that found in the actual population). Observe that

$$\Pr(D = 1 \mid J(Z) = j) = \frac{\int \int \mathbf{1}(\mu_D(z) \geq v) f_{\mu_D, J(Z)}(\mu_D, j) f_V(v) dv d\mu_D}{\int f_{\mu_D, J(Z)}(\mu_D, j) d\mu_D},$$

but this is only for the construction given by the rational expectations assumption.

## II The Yitzhaki Weights for OLS and 2SLS

This appendix explicates the Yitzhaki (1989, 1996), and Yitzhaki and Schechtman (2004) interpretation of the *OLS* weights and applies it to *2SLS*. We first develop some background results useful in computing the *OLS* and *2SLS* weights. The sample covariance is

$$\frac{1}{N} \sum_{i=1}^N (Y_i - \bar{Y})(X_i - \bar{X})$$

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where

$$\bar{Y} = \frac{1}{N} \sum_{i=1}^N Y_i \quad \bar{X} = \frac{1}{N} \sum_{i=1}^N X_i.$$

It can always be written in  $U$  statistic form as  $\frac{1}{N(N-1)} \sum_{1 \leq i < j \leq N} (X_j - X_i)(Y_j - Y_i)$ . For notational simplicity assume  $X_{i+1} > X_i$  for all  $i = 1, \dots, N-1$ . Thus the  $OLS$  estimator can be written as

$$\hat{\beta}_{OLS} = \frac{\sum_{1 \leq i < j \leq N} (X_j - X_i)(Y_j - Y_i)}{\sum_{1 \leq i < j \leq N} (X_j - X_i)^2}$$

Now, order the observations so that  $X_1 \leq X_2 \leq \dots \leq X_N$ . Focus on the numerator of this expression. Observe that for  $N > j \geq i \geq 1$ , we may, for fixed  $j$  write the contribution of term  $j$  to the expression as

$$\sum_{1 \leq i < j \leq N} (X_j - X_i)(Y_j - Y_i) = \sum_{1 \leq i < j \leq N} ((X_j - X_i)(Y_j - Y_{j-1} + Y_{j-1} - Y_{j-2} + Y_{j-2} - Y_{j-3} + \dots))$$

Let  $\Delta Y_{j-1} = Y_j - Y_{j-1}$ .

Then we can, form the pairwise  $OLS$  estimator, as in Theil (1950). Theil considers all pairwise  $OLS$  coefficients  $b_{ji} = \left( \frac{Y_j - Y_i}{X_j - X_i} \right) 1(X_j \neq X_i)$ . Yitzhaki forms all ordered slopes.

$$b_{j-1} = \left( \frac{Y_j - Y_{j-1}}{X_j - X_{j-1}} \right) 1(X_j \neq X_{j-1}).$$

i.e. the pairwise  $OLS$  slope on adjacent observations ordered by the  $X$ . Thus we can write the contribution of this term to the numerator as

$$\sum_{1 \leq i < j \leq N} (X_j - X_i)(b_{j-1} \Delta X_{j-1} + \dots + b_i \Delta X_i)$$

which is a double sum over  $i$  and  $j$ . Substituting the same decomposition for  $(X_j - X_i)$ , the numerator is

$$\sum_{1 \leq i < j \leq N} (X_j - X_{j-1} + X_{j-1} - X_{j-2} + \dots + X_{i+1} - X_i)(b_{j-1} \Delta X_{j-1} + \dots + b_i \Delta X_i).$$

A similar decomposition holds for the denominator where  $\Delta X_i$  is substituted for  $\Delta Y_i$  (and  $b_i = 1$  for all  $i$ ). Observe that if  $(X_i = X_{i'})$  or  $(Y_i = Y_{i'})$  we get no contribution from the comparison  $i$  and  $i'$ . (We can cover this case by using the indicator  $1(X_i \neq X_{i'})$  everywhere in the numerator.) The structure looks complex but has an interesting, simplifying pattern. To gain some intuition, write out the expression in full using the order on the  $X$ :

$$\begin{array}{llll} (X_N - X_1)(Y_N - Y_1) & = & [b_{N-1} \Delta X_{N-1} + & \dots + b_k \Delta X_k + \dots + b_2 \Delta X_2 + b_1 \Delta X_1] & \sum_{j=1}^{N-1} \Delta X_j \\ (X_N - X_2)(Y_N - Y_2) & = & [b_{N-1} \Delta X_{N-1} + & \dots + b_k \Delta X_k + \dots + b_2 \Delta X_2 & + 0] & \sum_{j=2}^{N-1} \Delta X_j \\ \dots & & & & & \\ (X_N - X_k)(Y_N - Y_k) & = & [b_{N-1} \Delta X_{N-1} + & \dots + b_k \Delta X_k + \dots & + 0 & + 0] & \sum_{j=k}^{N-1} \Delta X_j \\ \dots & & & & & \\ (X_N - X_{N-1})(Y_N - Y_{N-1}) & = & [b_{N-1} \Delta X_{N-1} + 0 + & \dots + 0 + \dots & + 0 & + 0] & \sum_{j=N-1}^{N-1} \Delta X_j \\ \dots & & & & & \\ (X_{N-1} - X_1)(Y_{N-1} - Y_1) & = & [0 + \dots & \dots + b_k \Delta X_k + \dots + b_2 \Delta X_2 + b_1 \Delta X_1] & \sum_{j=1}^{N-1} \Delta X_j \\ \dots & & & & & \\ (X_{N-1} - X_2)(Y_{N-1} - Y_2) & = & [0 + \dots & \dots + b_k \Delta X_k + \dots + b_2 \Delta X_2 & + 0] & \sum_{j=2}^{N-1} \Delta X_j \\ (X_{N-1} - X_k)(Y_{N-1} - Y_k) & = & [0 + \dots & \dots + b_k \Delta X_k + \dots & + 0 & + 0] & \sum_{j=k}^{N-1} \Delta X_j \end{array}$$

and so forth. Writing it this way we put the contribution of  $b_{N-1}$  in the first column; the contribution of  $b_{N-2}$  in the second column and so forth.

Going down columns and across blocks, the contribution of the  $b_{N-1}$  term to the numerator is

$$(b_{N-1} \Delta X_{N-1}) \sum_{l=1}^{N-1} \sum_{j=l}^{N-1} \Delta X_j.$$

Consider the second term

$$(b_{N-2}\Delta X_{N-2}) \left[ \sum_{l=1}^{N-2} \sum_{j=l}^{N-1} \Delta X_j + \sum_{l=1}^{N-2} \sum_{j=l}^{N-2} \Delta X_j \right].$$

The contribution of the  $b_{N-k}$  to the numerator is

$$[b_{N-k}\Delta X_{N-k}] \left[ \sum_{l=1}^{N-k} \sum_{j=l}^{N-1} \Delta X_j + \sum_{l=1}^{N-k} \sum_{j=l}^{N-2} \Delta X_j + \cdots + \sum_{l=1}^{N-k} \sum_{j=l}^{N-k} \Delta X_j \right].$$

The generic term in braces can be written out in the following way (see the following table obtained by fixing  $l = 1$ , then  $l = 2$  and so forth for each term).

$$(N-k) \text{ rows} \begin{array}{c} (N-1) \text{ cols} \\ \left[ \begin{array}{cccccc} \Delta X_{N-1} & +\Delta X_{N-2} & +\cdots & +\Delta X_{N-k} & +\cdots & +\Delta X_2 & +\Delta X_1 \\ \Delta X_{N-1} & +\Delta X_{N-2} & +\cdots & +\Delta X_{N-k} & +\cdots & +\Delta X_2 & +0 \\ & & & \vdots & & & \\ \Delta X_{N-1} & +\Delta X_{N-2} & +\cdots & +\Delta X_{N-k} & +\cdots & +0 & +0 \end{array} \right] \end{array}$$

corresponds to the first term;

$$(N-k) \text{ rows} \begin{array}{c} + \\ (N-1) \text{ cols} \\ \left[ \begin{array}{ccccc} 0+ & \Delta X_{N-2} & +\cdots & +\Delta X_2 & +\Delta X_1 \\ 0+ & \Delta X_{N-2} & +\cdots & +\Delta X_2 & +0 \\ & & \vdots & 0 & 0 \\ 0+ & \Delta X_{N-2} & +\cdots & 0 & 0 \end{array} \right] \end{array}$$

corresponds to the second term;

$$(N-k) \text{ rows} \begin{array}{c} + \\ \vdots \\ + \\ (N-1) \text{ cols} \\ \left[ \begin{array}{cccccc} 0+0+ & \cdots & +\Delta X_{N-k} & +\cdots & +\Delta X_3 & +\Delta X_2 & +\Delta X_1 \\ 0+0+ & \cdots & +\Delta X_{N-2} & +\cdots & +\Delta X_3 & +\Delta X_2 & 0 \\ & & \vdots & 0 & 0 & 0 & 0 \\ 0+0+ & \cdots & +\Delta X_{N-k} & +\cdots & 0 & 0 & 0 \end{array} \right] \end{array}$$

corresponds to the  $k$ th term.

Summing across all of the rows,

$$\begin{aligned} & (N-k)(\Delta X_{N-1}) + 2(N-k)\Delta X_{N-2} + 3(N-k)\Delta X_{N-3} + \cdots + (N-k)(N-k)\Delta X_{N-k} \\ & + k\Delta X_1 + 2k\Delta X_2 + \cdots + (N-k-1)\Delta X_{N-k-1} \\ = & (N-k) \sum_{l=1}^k l\Delta X_{N-l} + k \sum_{q=1}^{N-k-1} q\Delta X_q. \end{aligned}$$

Letting  $m = N - l$ , this expression can be written as

$$= (N-k) \sum_{m=N-k}^{N-1} (N-m)\Delta X_m + k \sum_{q=1}^{N-k-1} q\Delta X_q.$$

Let  $N - k = i$  and use the same index for  $m$  and  $q$ . The weight on  $b_i$  is then

$$i \sum_{j=i}^{N-1} (N-j) \Delta X_j + (N-i) \sum_{j=1}^{i-1} j \Delta X_j$$

which is the numerator of Yitzhaki's weight. Write out the expression for the  $i^{\text{th}}$  weight.

$$(i) [(N-i) \Delta X_i + (N-i-1) \Delta X_{i+1} + \dots + \Delta X_{N-1}] \\ + (N-i) [\Delta X_1 + 2\Delta X_2 + \dots + (i-1) \Delta X_{i-1}]. \quad (\text{Y-1})$$

Look at the second line

$$(N-i) [(X_2 - X_1) + 2(X_3 - X_2) + 3(X_4 - X_3) + \dots + (i-2)(X_{i-1} - X_{i-2}) + (i-1)(X_i - X_{i-1})] \\ = (N-1)[-X_1 - X_2 - \dots - X_{i-1} - X_i] + iX_i(N-i).$$

Adding the two lines and collecting terms on  $i$  we obtain

$$(Ni) (\bar{X}_N - \bar{X}_i)$$

where

$$\bar{X}_i = \frac{1}{i} \sum_{j=1}^i X_j \quad \text{and} \quad \bar{X}_N = \frac{1}{N} \sum_{j=1}^N X_j.$$

Therefore the weight is positive. The denominator is  $N\sigma_X^2$ . Thus the weight can be written (for the  $i$ th ordered regression coefficient) as

$$\frac{i(\bar{X}_N - \bar{X}_i)}{\sigma_X^2}$$

( $\sigma_X^2$  is divided by  $N$ ).  $i$  is the (discrete) quantile of the  $N$  observations. This form is the same form as the expression for  $P(P(Z) \geq u_D)$  in the weights as presented in the text.

An alternative form of the Yitzhaki weight notes that adding the two lines of (Y-1) we obtain

$$-(N-i) \sum_{j=1}^N X_j + N \sum_{j=i+1}^N X_j \\ = N \left[ \sum_{j=i+1}^N X_j - (N-i) \sum_{j=1}^N \left( \frac{X_j}{N} \right) \right] \\ = (N-i)(N) \left[ \sum_{j=i+1}^N \left( \frac{X_j - \bar{X}}{N-i} \right) \right]$$

which is the form of the  $IV$  weight developed in the text, and can be written as

$$\omega_i = \frac{(N-i)}{\sigma_X^2} \sum_{j=i+1}^N \left( \frac{X_j - \bar{X}}{N-i} \right) \\ = \frac{(N-i)}{\hat{\sigma}_X^2} E(X - \bar{X} \mid X > X_i)$$

where  $E$  denotes the sample mean of  $X$  above  $\bar{X}$ .

## Extension to TSLS

Replacing  $X$  by  $E(X \mid Z)$  where  $Z$  is the instrument, the extension to is  $TSLS$  immediate.

### III Derivation of the Weights for the Mixture of Normals Example

Using the notation defined in the text in Section 5.2, and writing  $E_1$  as the expectation for group 1, letting  $\mu_1$  be the mean of  $Z$  for population 1 and  $\mu_{11}$  be the mean of the first component of  $Z$ ,

$$\begin{aligned} E_1(Z_1 | \gamma'Z > v) &= \mu_{11} + \frac{\gamma'\Sigma_1^1}{\gamma'\Sigma_1\gamma} E_1(Z_1 - \mu_1 | \gamma'Z > v) \\ &= \mu_{11} + \frac{\gamma'\Sigma_1^1}{(\gamma'\Sigma_1\gamma)^{1/2}} E_1\left(\frac{\gamma'(Z - \mu_1)}{(\gamma'\Sigma_1\gamma)^{1/2}} \mid \frac{\gamma'(Z - \mu_1)}{(\gamma'\Sigma_1\gamma)^{1/2}} > \frac{(v - \gamma'\mu_1)}{(\gamma'\Sigma_1\gamma)^{1/2}}\right) \\ &= \mu_{11} + \frac{\gamma'\Sigma_1^1}{(\gamma'\Sigma_1\gamma)^{1/2}} \lambda\left(\frac{(v - \gamma'\mu_1)}{(\gamma'\Sigma_1\gamma)^{1/2}}\right) \end{aligned}$$

where

$$\lambda(c) = \frac{1}{\sqrt{2\pi}} \frac{e^{-c^2/2}}{\Phi(-c)}$$

where  $\Phi(\cdot)$  is the unit normal cumulative distribution function.

By the same logic, in the second group:

$$E_2(Z_1 | \gamma'Z > v) = \mu_{21} + \frac{\gamma'\Sigma_2^1}{(\gamma'\Sigma_2\gamma)^{1/2}} \lambda\left(\frac{(v - \gamma'\mu_2)}{(\gamma'\Sigma_2\gamma)^{1/2}}\right)$$

Therefore for the overall population we obtain

$$\begin{aligned} &E(Z_1 - E(Z_1) | \gamma'Z > v) \Pr(\gamma'Z > v) \\ &= (P_1 \Pr_1(\gamma'Z > v) \mu_{11} + P_2 \Pr_2(\gamma'Z > v) \mu_{21}) + \frac{P_1 \gamma \Sigma_1^1}{(\gamma' \Sigma_1 \gamma)^{1/2} \sqrt{2\pi}} \exp\left[-\frac{1}{2} \left(\frac{v - \gamma' \mu_1}{(\gamma' \Sigma_1 \gamma)^{1/2}}\right)^2\right] \\ &\quad + \frac{P_2 \gamma \Sigma_2^1}{(\gamma' \Sigma_2 \gamma)^{1/2} \sqrt{2\pi}} \exp\left[-\frac{1}{2} \left(\frac{v - \gamma' \mu_2}{(\gamma' \Sigma_2 \gamma)^{1/2}}\right)^2\right] - (P_1 \mu_{11} + P_2 \mu_{21}) \Pr(\gamma'Z > v) \\ &= P_1 \mu_{11} (\Pr_1(\gamma'Z > v) - \Pr(\gamma'Z > v)) + P_2 \mu_{21} (\Pr_2(\gamma'Z > v) - \Pr(\gamma'Z > v)) + \\ &\quad \frac{P_1 \gamma \Sigma_1^1}{(\gamma' \Sigma_1 \gamma)^{1/2} \sqrt{2\pi}} \exp\left[-\frac{1}{2} \left(\frac{v - \gamma' \mu_1}{(\gamma' \Sigma_1 \gamma)^{1/2}}\right)^2\right] + \frac{P_2 \gamma \Sigma_2^1}{(\gamma' \Sigma_2 \gamma)^{1/2} \sqrt{2\pi}} \exp\left[-\frac{1}{2} \left(\frac{v - \gamma' \mu_2}{(\gamma' \Sigma_2 \gamma)^{1/2}}\right)^2\right] \\ &= P_1 P_2 (\mu_{11} - \mu_{21}) (\Pr_1(\gamma'Z > v) - \Pr_2(\gamma'Z > v)) + \\ &\quad \frac{P_1 \gamma \Sigma_1^1}{(\gamma' \Sigma_1 \gamma)^{1/2} \sqrt{2\pi}} \exp\left[-\frac{1}{2} \left(\frac{v - \gamma' \mu_1}{(\gamma' \Sigma_1 \gamma)^{1/2}}\right)^2\right] + \frac{P_2 \gamma \Sigma_2^1}{(\gamma' \Sigma_2 \gamma)^{1/2} \sqrt{2\pi}} \exp\left[-\frac{1}{2} \left(\frac{v - \gamma' \mu_2}{(\gamma' \Sigma_2 \gamma)^{1/2}}\right)^2\right] \end{aligned}$$

where  $\Pr_1(\gamma'Z > v)$  and  $\Pr_2(\gamma'Z > v)$  represent the probability of the event  $\gamma'Z > v$  in sub-population 1 and 2, respectively. The last equality follows from  $\Pr(\gamma'Z > v) = P_1 \Pr_1(\gamma'Z > v) + P_2 \Pr_2(\gamma'Z > v)$ .

We need  $Cov(D, Z_1)$ . To obtain it, observe that

$$\begin{aligned} D &= \mathbf{1}[\gamma'Z - V > 0] \\ E(Z_1, D) &= E(Z_1 \mathbf{1}(\gamma'Z - V > 0)). \end{aligned}$$

Let  $E_1$  denote the expectation under Group 1, and let  $E_2$  denote the expectation under Group 2.

$$E(Z_1 D) = E(Z_1 | \gamma'Z - V > 0) \Pr(\gamma'Z - V > 0)$$

$$\begin{aligned}
E(Z_1 D) &= \left\{ P_1 \left[ \mu_{11} \Pr_1(\gamma'Z - V > 0) + \frac{\gamma' \Sigma_1^1}{\gamma' \Sigma_1 \gamma + \sigma_V^2} \lambda \left( \frac{-\gamma' \mu_1}{(\gamma' \Sigma_1 \gamma + \sigma_V^2)^{1/2}} \right) \right] \right. \\
&\quad \left. + P_2 \left[ \mu_{21} \Pr_2(\gamma'Z - V > 0) + \frac{\gamma' \Sigma_2^1}{\gamma' \Sigma_2 \gamma + \sigma_V^2} \lambda \left( \frac{-\gamma' \mu_2}{(\gamma' \Sigma_2 \gamma + \sigma_V^2)^{1/2}} \right) \right] \right\} \\
&= (P_1 \Pr_1(\gamma'Z > V) \mu_{11} + P_2 \Pr_2(\gamma'Z > V) \mu_{21}) \\
&\quad + \frac{P_1 \gamma' \Sigma_1^1}{(\gamma' \Sigma_1 \gamma + \sigma_V^2)^{1/2} \sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \frac{-\gamma' \mu_1}{(\gamma' \Sigma_1 \gamma + \sigma_V^2)^{1/2}} \right)^2 \right] \\
&\quad + \frac{P_2 \gamma' \Sigma_2^1}{(\gamma' \Sigma_2 \gamma + \sigma_V^2)^{1/2} \sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \frac{-\gamma' \mu_2}{(\gamma' \Sigma_2 \gamma + \sigma_V^2)^{1/2}} \right)^2 \right].
\end{aligned}$$

Because

$$E(D)E(Z_1) = \Pr(\gamma'Z - V > 0) (P_1 \mu_{11} + P_2 \mu_{21})$$

and

$$\text{Cov}(D, Z_1) = E(Z_1 D) - E(Z_1)E(D)$$

$$\begin{aligned}
\therefore \text{Cov}(D, Z_1) &= P_1 \mu_{11} (\Pr_1(\gamma'Z > V) - \Pr(\gamma'Z > V)) - P_2 \mu_{21} (\Pr_2(\gamma'Z > V) - \Pr(\gamma'Z > V)) \\
&\quad + \frac{P_1 \gamma' \Sigma_1^1}{(\gamma' \Sigma_1 \gamma + \sigma_V^2)^{1/2} \sqrt{2\pi}} \exp \left[ -\left( \frac{-\gamma' \mu_1}{(\gamma' \Sigma_1 \gamma + \sigma_V^2)^{1/2}} \right)^2 \right] \\
&\quad + \frac{P_2 \gamma' \Sigma_2^1}{(\gamma' \Sigma_2 \gamma + \sigma_V^2)^{1/2} \sqrt{2\pi}} \exp \left[ -\left( \frac{-\gamma' \mu_2}{(\gamma' \Sigma_2 \gamma + \sigma_V^2)^{1/2}} \right)^2 \right].
\end{aligned}$$

Thus the  $IV$  weights for this set-up are:

$$\tilde{\omega}_{IV}(v) = \frac{\left\{ \begin{aligned} &P_1 P_2 (\mu_{11} - \mu_{21}) (\Pr_1(\gamma'Z > v) - \Pr_2(\gamma'Z > v)) \\ &+ \left[ \frac{P_1 \gamma' \Sigma_1^1}{(\gamma' \Sigma_1 \gamma + \sigma_V^2)^{1/2} \sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \frac{v - \gamma' \mu_1}{(\gamma' \Sigma_1 \gamma + \sigma_V^2)^{1/2}} \right)^2 \right] + \frac{P_2 \gamma' \Sigma_2^1}{(\gamma' \Sigma_2 \gamma + \sigma_V^2)^{1/2} \sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \frac{v - \gamma' \mu_2}{(\gamma' \Sigma_2 \gamma + \sigma_V^2)^{1/2}} \right)^2 \right] \right] f_V(v) \end{aligned} \right\}}{\left\{ \begin{aligned} &P_1 P_2 (\mu_{11} - \mu_{21}) (\Pr_1(\gamma'Z > V) - \Pr_2(\gamma'Z > V)) \\ &+ \left[ \frac{P_1 \gamma' \Sigma_1^1}{(\gamma' \Sigma_1 \gamma + \sigma_V^2)^{1/2} \sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \frac{-\gamma' \mu_1}{(\gamma' \Sigma_1 \gamma + \sigma_V^2)^{1/2}} \right)^2 \right] + \frac{P_2 \gamma' \Sigma_2^1}{(\gamma' \Sigma_2 \gamma + \sigma_V^2)^{1/2} \sqrt{2\pi}} \exp \left[ -\frac{1}{2} \left( \frac{-\gamma' \mu_2}{(\gamma' \Sigma_2 \gamma + \sigma_V^2)^{1/2}} \right)^2 \right] \right] \end{aligned} \right\}}.$$

where  $\sigma_V^2$  represents the variance of  $V$ . Clearly,  $\tilde{\omega}_{IV}(-\infty) = 0$ ,  $\tilde{\omega}_{IV}(\infty) = 0$  and the weights integrate to one over the support of  $V = (-\infty, \infty)$ . Observe that the weights must be positive if  $P_2 = 0$ . Thus the structure of the covariances of the instrument with the choice index  $\gamma'Z$  is a key determinant of the positivity of the weights for any instrument. It has nothing to do with the *ceteris paribus* effect of  $Z_1$  on  $\gamma'Z$  or  $P(Z)$  in the general case.

In order to simplify the analysis and illustrate the main ideas of this example, we assume  $\mu_{11} = \mu_{21}$  in what follows.<sup>1</sup> In this case, a necessary condition for  $\omega_{IV} < 0$  over some values of  $v$  is that  $\text{sign}(\gamma' \Sigma_1^1) = -\text{sign}(\gamma' \Sigma_2^1)$ , *i.e.*, that the covariance between  $Z_1$  and  $\gamma'Z$  be of opposite signs in the two subpopulations so  $Z_1$  and  $P(Z)$  have different relationships in the two component populations. Without loss of generality assume that  $\gamma' \Sigma_1^1 > 0$ . If it

<sup>1</sup>This is the case of two of our examples in the paper.

equals zero, we fail the rank condition in the first population and we are back to a one subpopulation model with positive weights. The numerator of the expression for  $\omega_{IV}(v)$  switches signs if for some values of  $v$ ,

$$\frac{P_1 \gamma' \Sigma_1^1}{(\gamma' \Sigma_1 \gamma)^{1/2}} \exp \left[ -\frac{1}{2} \left( \frac{v - \gamma' \mu_1}{(\gamma' \Sigma_1 \gamma)^{1/2}} \right)^2 \right] < -\frac{P_2 \gamma' \Sigma_2^1}{(\gamma' \Sigma_2 \gamma)^{1/2}} \exp \left[ -\frac{1}{2} \left( \frac{v - \gamma' \mu_2}{(\gamma' \Sigma_2 \gamma)^{1/2}} \right)^2 \right]$$

while for other values the inequality is reversed. (Observe that the denominator is a constant.) Rewriting and taking logarithms, we obtain under the assumption that  $\text{sign}(\gamma' \Sigma_1^1) = -\text{sign}(\gamma' \Sigma_2^1)$ , the following expression:

$$\frac{1}{2} \left[ \frac{(v - \gamma' \mu_2)^2}{\gamma' \Sigma_2 \gamma} - \frac{(v - \gamma' \mu_1)^2}{\gamma' \Sigma_1 \gamma} \right] < \ln \left( \frac{1 - P_1}{P_1} \right) + \ln \left[ \frac{-\gamma' \Sigma_2^1}{\gamma' \Sigma_1^1} \right] + \ln \left[ \frac{\gamma' \Sigma_1 \gamma}{\gamma' \Sigma_2 \gamma} \right]$$

where we assume  $0 < P_1 < 1$ . Observe that  $\frac{1 - P_1}{P_1}$  can be made as large or as small as we like by varying  $P_1$ . Varying  $(\mu_1, \mu_2)$  does not affect the right hand side. For  $\mu_1 = \mu_2 = 0$ , the inequality becomes

$$\frac{1}{2} v^2 \left[ \frac{1}{\gamma' \Sigma_2 \gamma} - \frac{1}{\gamma' \Sigma_1 \gamma} \right] < \ln \left( \frac{1 - P_1}{P_1} \right) + \ln \left[ \frac{-\gamma' \Sigma_2^1}{\gamma' \Sigma_1^1} \right] + \ln \left[ \frac{\gamma' \Sigma_1 \gamma}{\gamma' \Sigma_2 \gamma} \right].$$

Suppose that  $\gamma' \Sigma_2 \gamma < \gamma' \Sigma_1 \gamma$ . Then the left hand side is positive except when  $v = 0$ . For any fixed  $\gamma, \Sigma_1, \Sigma_2$  we can find a value of  $P_1$  sufficiently small so that right hand side of the equation is positive and for any such value of  $P_1$  there will be a  $v$  sufficiently small for the inequality to be satisfied. There is also a value of  $v$  that reverses the inequality.

The inequality is satisfied for some  $v^* \geq 0$ . But with  $v$  arbitrarily large, the inequality can be reversed so that the weight will switch signs at some value of  $v$ . The key necessary condition is that  $Cov(Z_1, \gamma' Z)$  be of opposite signs in the two subpopulations. Using  $Z_1$  as an *IV*, but not conditioning or controlling for the other components of  $Z$ , produces sometimes negative and sometimes positive movements in the components of  $Z_2, \dots, Z_k$  which can offset the *ceteris paribus* ( $Z_2 = z_2, \dots, Z_k = z_k$ ) movements of  $Z_1$ .

## IV Extensions of the Unordered Case

The argument in the text shows that *IV* using a variable that shifts people toward (or against)  $j$  by operating only on  $R_j(Z_j)$  can identify the mean marginal return to  $j$  vs. the next best alternative, which may be different for different people. The analysis of the binary case extends directly to this case.

However, without further assumptions, *IV* will not decompose the marginal return into its component parts. This has implications for interpreting an *IV* analysis of the effect of “schooling,”  $S = \sum_{j \in \mathcal{J}} j D_{\mathcal{J},j}$ , on earnings. To see this, consider a three outcome case. Our result is more general but the three outcome case is easy to exposit.

Thus suppose we seek to interpret what

$$\Delta_{Z_1}^{IV} = \frac{Cov(Z_1, Y)}{Cov(Z_1, S)} \tag{1}$$

estimates in terms of the decompositions like those presented in (3.7)-(3.8) in the text and in (6.15)-(6.16).

We seek a decomposition that is

1. Recoverable by *LIV* or *LATE*
2. The weights can be estimated from the data;
3. Economically interpretable.

Recall that the Imbens-Angrist measure in their 1995 paper discussed in the introduction to Section 6 fails criteria 1 and 3.

Take the three outcome case:

$$R_i = \vartheta_i(Z_i) - V_i, \quad i = 1, 2, 3,$$

$$Z = (Z_1, Z_2, Z_3) \perp\!\!\!\perp (V_1, V_2, V_3, Y_1, Y_2, Y_3).$$

We keep conditioning on  $X$  implicit. Suppose  $Z_1, Z_2, Z_3$  are disjoint sets of regressors  $Z = (Z_1, Z_2, Z_3)$  (not necessarily statistically independent). We can easily relax this assumption but it is helpful to assume this case in making basic points. It simplifies the notation in the text. We can always condition on common  $Z$  components. In this notation

$$\begin{aligned} E(Y | Z) &= E \left[ \sum_{s=1}^3 Y_s D_s \middle| Z \right] \\ &= E(Y_1 D_1 | Z) + E(Y_2 D_2 | Z) + E(Y_3 D_3 | Z) \\ D_1 &= \mathbf{1}(R_1 \geq R_2, R_1 \geq R_3) \\ D_2 &= \mathbf{1}(R_2 \geq R_1, R_2 \geq R_3) \\ D_3 &= \mathbf{1}(R_3 \geq R_1, R_3 \geq R_2). \end{aligned}$$

$$\begin{aligned} \frac{\partial E(Y_1 D_1 | Z)}{\partial Z_1} &= \frac{\partial}{\partial Z_1} \int \int_{-\infty}^{\vartheta_1(Z_1) - \vartheta_2(Z_2)} \int_{-\infty}^{\vartheta_1(Z_1) - \vartheta_3(Z_3)} y_1 f(y_1, v_1 - v_2, v_1 - v_3) dy_1 d(v_1 - v_2) d(v_1 - v_3) \\ &= \frac{\partial \vartheta_1(Z_1)}{\partial Z_1} \left[ \int y_1 \int_{-\infty}^{\vartheta_1(Z_1) - \vartheta_3(Z_3)} f(y_1, \vartheta_1(Z_1) - \vartheta_2(Z_2), v_1 - v_3) dy_1 d(v_1 - v_3) \right. \\ &\quad \left. + \int y_1 \int_{-\infty}^{\vartheta_1(Z_1) - \vartheta_2(Z_2)} f(y_1, v_1 - v_2, \vartheta_1(Z_1) - \vartheta_3(Z_3)) dy_1 d(v_1 - v_2) \right]. \end{aligned} \quad (2)$$

We can recover this combination of parameters from the data on  $Y_1 D_1$ .

$$\begin{aligned} \frac{\partial E(Y_2 D_2 | Z)}{\partial Z_1} &= \frac{\partial}{\partial Z_1} \int y_2 \int_{-\infty}^{\vartheta_2(Z_2) - \vartheta_1(Z_1)} \int_{-\infty}^{\vartheta_2(Z_2) - \vartheta_3(Z_3)} f(y_2, v_2 - v_1, v_2 - v_3) d(v_1 - v_2) d(v_2 - v_3) dy_2 \\ &= \frac{-\partial \vartheta_1(Z_1)}{\partial Z_1} \left[ \int y_2 \int_{-\infty}^{\vartheta_2(Z_2) - \vartheta_3(Z_3)} f(y_2, \vartheta_2(Z_2) - \vartheta_1(Z_1), v_2 - v_3) dy_2 d(v_2 - v_3) \right]. \end{aligned} \quad (3)$$

We can recover this combination of parameters from the data on  $Y_3 D_3$ .

$$\frac{\partial E(Y_3 D_3 | Z)}{\partial Z_1} = \frac{-\partial \vartheta_1(Z_1)}{\partial Z_1} \int y_3 \int_{-\infty}^{\vartheta_3(Z_3) - \vartheta_2(Z_2)} f(y_3, \vartheta_3(Z_3) - \vartheta_1(Z_1), v_3 - v_2) dy_3 d(v_3 - v_2) \quad (4)$$

We can recover these combinations of parameters from the data. Notice that “ $f$ ” is an abstract expression with different meaning in (2)–(4). The first term can be decomposed into two conditioning sets corresponding to expression (2):

$$\begin{aligned} (R_1 = R_2) \wedge (R_1 \geq R_3) &\quad (\text{first part}), \\ (R_1 = R_3) \wedge (R_1 \geq R_2) &\quad (\text{second part}). \end{aligned}$$

Decomposing it into components is not possible without invoking additional assumptions (e.g. large support conditions as in Heckman and Vytlacil, 2006a,b, that send some of the components of  $\vartheta_l(Z_l)$ ,  $l = 1, 2, 3$ , off to  $\pm\infty$ ). The second expression corresponds to the conditioning set

$$(R_1 = R_2) \wedge (R_2 \geq R_3) \Leftrightarrow (R_1 = R_2) \wedge (R_1 \geq R_3).$$

The third term corresponds to the conditioning set

$$(R_1 = R_3) \wedge (R_3 \geq R_2) \Leftrightarrow (R_1 = R_3) \wedge (R_1 \geq R_2).$$

If we differentiate with respect to a component that is specialized to  $Z_1$ , we obtain

$$\frac{\left( \frac{\partial E(Y|Z)}{\partial Z_1} \right)}{\left( \frac{\partial \vartheta_1}{\partial Z_1} \right)} = \left[ \begin{array}{l} [E(Y_1 - Y_2 | R_1(Z_1) = R_2(Z_2), R_1(Z_1) \geq R_3(Z_3))] f(R_1(Z_1) = R_2(Z_2), R_1(Z_1) \geq R_3(Z_3)) \\ + [E(Y_1 - Y_3 | R_1(Z_1) = R_3(Z_3), R_1(Z_1) \geq R_2(Z_2))] f(R_1(Z_1) = R_3(Z_3), R_1(Z_1) \geq R_2(Z_2)) \end{array} \right]$$

This is the return at margin. It comes from two sources: persons who come from (or exit) 2 and persons who come from (or exit) 3.

By the chain rule, we can express this in terms of the derivative of the  $\Pr(D_1 = 1 | Z)$ :

$$\frac{\partial \Pr(D_1 = 1 | Z)}{\partial Z_1} = \left( \frac{\partial \vartheta_1}{\partial Z_1} \right) \left[ \begin{array}{l} f(R_1(Z_1) = R_2(Z_2), R_1(Z_1) \geq R_3(Z_3)) \\ + f(R_1(Z_1) = R_3(Z_3), R_1(Z_1) \geq R_2(Z_2)) \end{array} \right].$$

Thus we can write  $LIV$  using  $\Pr(D_1 = 1 | Z)$  as:

$$LIV = \frac{\left( \frac{\partial E(Y|Z)}{\partial Z_1} \right)}{\left( \frac{\partial \Pr(D_1=1|Z)}{\partial Z_1} \right)} = \left[ \begin{array}{l} E(Y_1 - Y_2 | R_1(Z_1) = R_2(Z_2), R_1(Z_1) \geq R_3(Z_3)) \omega_{12} \\ + E(Y_1 - Y_3 | R_1(Z_1) = R_3(Z_3), R_1(Z_1) \geq R_2(Z_2)) \omega_{13} \end{array} \right] \quad (5)$$

The weights are:

$$\omega_{12} = \frac{f(R_1(Z_1) = R_2(Z_2), R_1(Z_1) \geq R_3(Z_3))}{\left[ \begin{array}{l} f(R_1(Z_1) = R_2(Z_2), R_1(Z_1) \geq R_3(Z_3)) \\ + f(R_1(Z_1) = R_3(Z_3), R_1(Z_1) \geq R_2(Z_2)) \end{array} \right]} \quad (6)$$

$$\omega_{13} = \frac{f(R_1(Z_1) = R_3(Z_3), R_1(Z_1) \geq R_2(Z_2))}{\left[ \begin{array}{l} f(R_1(Z_1) = R_2(Z_2), R_1(Z_1) \geq R_3(Z_3)) \\ + f(R_1(Z_1) = R_3(Z_3), R_1(Z_1) \geq R_2(Z_2)) \end{array} \right]} \quad (7)$$

These weights can be constructed from a discrete choice analysis conditional on  $Z$  coupled with the distribution of  $Z$ . Identifying the component parts by  $IV$ ,

$$E(Y_1 - Y_2 | R_1(Z_1) = R_2(Z_2), R_1(Z_1) \geq R_3(Z_3))$$

or

$$E(Y_1 - Y_3 | R_1(Z_1) = R_3(Z_3), R_1(Z_1) \geq R_2(Z_2)),$$

is not possible in general. For a general  $Z$ , we pick up many other flows in other directions,  $1 \rightarrow 2$ ,  $2 \rightarrow 1$ .

- $Z \uparrow$  may cause two way flows depending on how  $R_1(Z_1), R_2(Z_2), R_3(Z_3)$  change.  $Z_1$  is assumed only to affect  $R_1(Z_1)$ .

Monotonicity here means the same pattern of flow is experienced by everyone. Our definition in the general case:

- $Z_j \uparrow \Rightarrow i \rightarrow j$  but not  $j \rightarrow i$ .  $\therefore$  we get a pattern such that everyone has the same pattern of movement. (This is uniformity.)

## What does standard $IV$ estimate?

Using  $\tilde{Z}_1 = Z_1 - \bar{Z}_1$ ,

$$\begin{aligned} Cov(Y, Z_1) &= E\left(\tilde{Z}_1 (Y_1 D_1 + Y_2 D_2 + Y_3 D_3)\right) \\ &\quad \text{Using } D_1 = 1 - D_2 - D_3 \text{ we obtain} \\ &= E\left(\tilde{Z}_1 (Y_1 + (Y_2 - Y_1) D_2 + (Y_3 - Y_1) D_3)\right) \\ &= E\left(\tilde{Z}_1 Y_1\right) + E\left(\tilde{Z}_1 (Y_2 - Y_1) D_2\right) + E\left(\tilde{Z}_1 (Y_3 - Y_1) D_3\right), \end{aligned}$$

where  $E\left(\tilde{Z}_1 Y_1\right) = 0$ . It is natural to decompose using choice "1" as the base, because  $Z_1$  affects only  $R_1(Z_1)$ . The remaining terms are

$$\begin{aligned} Cov(Y, Z_1) &= E\left(\tilde{Z}_1 (Y_2 - Y_1) \mathbf{1}(R_2(Z_2) \geq R_1(Z_1), R_2(Z_2) \geq R_3(Z_3))\right) \\ &\quad + E\left(\tilde{Z}_1 (Y_3 - Y_1) \mathbf{1}(R_3(Z_3) \geq R_1(Z_1), R_3(Z_3) \geq R_2(Z_2))\right) \\ &= E\left[\tilde{Z}_1 (Y_2 - Y_1) \mathbf{1}\left(\begin{array}{l} (\vartheta_2(Z_2) - \vartheta_1(Z_1) \geq V_2 - V_1) \\ \wedge (\vartheta_2(Z_2) - \vartheta_3(Z_3) \geq V_2 - V_3) \end{array}\right)\right] \\ &\quad + E\left[\tilde{Z}_1 (Y_3 - Y_1) \mathbf{1}\left(\begin{array}{l} (\vartheta_3(Z_3) - \vartheta_1(Z_1) \geq V_3 - V_1) \\ \wedge (\vartheta_3(Z_3) - \vartheta_2(Z_2) \geq V_3 - V_2) \end{array}\right)\right] \end{aligned}$$

$$\begin{aligned}
&= \int \tilde{z}_1 (\mu_2 - \mu_1 + u_2 - u_1) \\
&\quad \int_{-\infty}^{\vartheta_2(z_2) - \vartheta_1(z_1)} \int_{-\infty}^{\vartheta_2(z_2) - \vartheta_3(z_3)} f_{U_2 - U_1, V_2 - V_1, V_2 - V_3} (u_2 - u_1, v_2 - v_1, v_2 - v_3) \\
&\quad d(u_2 - u_1) d(v_2 - v_1) d(v_2 - v_3) \\
&\quad f(\tilde{z}_1, \vartheta_2(z_2) - \vartheta_1(z_1), \vartheta_2(z_2) - \vartheta_3(z_3)) \\
&\quad d\tilde{z}_1 d(\vartheta_2(z_2) - \vartheta_1(z_1)) d(\vartheta_2(z_2) - \vartheta_3(z_3)) \\
&+ \int \tilde{z}_1 (\mu_3 - \mu_1 + u_3 - u_1) \\
&\quad \int_{-\infty}^{\vartheta_3(z_3) - \vartheta_1(z_1)} \int_{-\infty}^{\vartheta_3(z_3) - \vartheta_2(z_2)} f_{U_3 - U_1, V_3 - V_1, V_3 - V_2} (u_3 - u_1, v_3 - v_1, v_3 - v_2) \\
&\quad d(u_3 - u_1) d(v_3 - v_1) d(v_3 - v_2) \\
&\quad \times f(\tilde{z}_1, \vartheta_3(z_3) - \vartheta_1(z_1), \vartheta_3(z_3) - \vartheta_2(z_2)) \\
&\quad d\tilde{z}_1 d(\vartheta_3(z_3) - \vartheta_1(z_1)) d(\vartheta_3(z_3) - \vartheta_2(z_2)).
\end{aligned}$$

We obtain the weights by first analyzing the numerator term of  $\Delta_{Z_1}^{IV}$  in (1):

$$\begin{aligned}
&\int_{-\infty}^{\infty} E(Y_2 - Y_1 \mid V_2 - V_1 = v_2 - v_1, \vartheta_2(Z_2) - \vartheta_3(Z_3) \geq V_2 - V_3) \\
&\left\{ \int_{-\infty}^{\infty} \tilde{z}_1 \int_{-\infty}^{\infty} \left[ \left( \int_{-\infty}^{\vartheta_2(z_2) - \vartheta_3(z_3)} h_{V_2 - V_1, V_2 - V_3} (v_2 - v_1, v_2 - v_3) d(v_2 - v_3) \right) \right. \right. \\
&\quad \left. \left. \times \left( \int_{v_2 - v_1}^{\infty} g(\tilde{z}_1, \vartheta_2(z_2) - \vartheta_1(z_1), \vartheta_2(z_2) - \vartheta_3(z_3)) d(\vartheta_2(z_2) - \vartheta_1(z_1)) \right) \right] \right. \\
&\quad \left. d(\vartheta_2(z_2) - \vartheta_3(z_3)) d\tilde{z}_1 \right\} d(v_2 - v_1). \tag{8}
\end{aligned}$$

Define the term in braces in braces in (8) as  $\eta(Z_1, v_2 - v_1)$ . The expression in braces is our weight. It can be estimated from discrete choice analysis conditional on  $Z$  along with the joint distribution of  $Z$ . However, the term multiplying the weight cannot be identified by *LATE* or *LIV*. Thus this decomposition is not operational. By a parallel analysis,

$$\begin{aligned}
&\int_{-\infty}^{\infty} E(Y_3 - Y_1 \mid V_3 - V_1 = v_3 - v_1, \vartheta_3(Z_3) - \vartheta_2(Z_2) \geq V_3 - V_2) \\
&\left\{ \int_{-\infty}^{\infty} \tilde{z}_1 \int_{-\infty}^{\infty} \left[ \left( \int_{-\infty}^{\vartheta_3(z_3) - \vartheta_2(z_2)} h_{V_3 - V_1, V_3 - V_2} (v_3 - v_1, v_3 - v_2) d(v_3 - v_2) \right) \right. \right. \\
&\quad \left. \left. \times \left( \int_{v_3 - v_1}^{\infty} g(\tilde{z}_1, \vartheta_3(z_3) - \vartheta_1(z_1), \vartheta_3(z_3) - \vartheta_2(z_2)) d(\vartheta_3(z_3) - \vartheta_1(z_1)) \right) \right] \right. \\
&\quad \left. d(\vartheta_3(z_3) - \vartheta_2(z_2)) d\tilde{z}_1 \right\} d(v_3 - v_1). \tag{9}
\end{aligned}$$

Define the term in braces in braces in (9) as  $\eta(Z_1, v_3 - v_1)$ . To derive the denominator, recall that  $S = \sum_{s=1}^3 s D_s$ . Substitute again for  $D_1 = 1 - D_2 - D_3$ ,

$$\begin{aligned} \sum_{s=1}^3 s D_s &= (1 - D_2 - D_3) + 2D_2 + 3D_3 \\ &= 1 + D_2 + 2D_3, \\ \text{Cov}(S, \tilde{Z}_1) &= E\left(\tilde{Z}_1 D_2\right) + 2D\left(\tilde{Z}_1 D_3\right) \\ &= E\left(\tilde{Z}_1 (\mathbf{1}(R_2 \geq R_1, R_2 \geq R_3))\right) \\ &\quad + 2E\left(\tilde{Z}_1 (\mathbf{1}(R_3 \geq R_1, R_3 \geq R_2))\right). \end{aligned}$$

We get parallel expressions for the two terms corresponding to the two terms of (8) and (9). We obtain for the first term:

$$\begin{aligned} &\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{z}_1 \left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\tilde{z}_1, \vartheta_2(z_2) - \vartheta_1(z_1), \vartheta_2(z_2) - \vartheta_3(z_3)) \right. \\ &\quad \left. \int_{-\infty}^{\vartheta_2(z_2) - \vartheta_3(z_3)} h_{V_2 - V_1, V_2 - V_3}(v_2 - v_1, v_2 - v_3) d(v_2 - v_3) \right. \\ &\quad \left. \cdot d(\vartheta_2(z_2) - \vartheta_1(z_1)) d(\vartheta_2(z_2) - \vartheta_3(z_3)) \right] d\tilde{z}_1 d(v_2 - v_1). \end{aligned} \quad (10)$$

By Fubini's Theorem we obtain:

$$\begin{aligned} &\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{z}_1 \left[ \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\vartheta_2(z_2) - \vartheta_3(z_3)} h_{V_2 - V_1, V_2 - V_3}(v_2 - v_1, v_2 - v_3) d(v_2 - v_3) \right) \right. \\ &\quad \left. \cdot \left( \int_{v_2 - v_1}^{\infty} g(\tilde{z}_1, \vartheta_2(z_2) - \vartheta_1(z_1), \vartheta_2(z_2) - \vartheta_3(z_3)) d(\vartheta_2(z_2) - \vartheta_1(z_1)) \right) \right. \\ &\quad \left. \cdot d(\vartheta_2(z_2) - \vartheta_3(z_3)) \right] d\tilde{z}_1 d(v_2 - v_1) \\ &= \int_{-\infty}^{\infty} \eta(Z_1, v_2 - v_1) d(v_2 - v_1). \end{aligned} \quad (11)$$

By parallel logic, we obtain for the second term:

$$\begin{aligned} &\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{z}_1 \left[ \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\vartheta_3(z_3) - \vartheta_2(z_2)} h_{V_3 - V_1, V_3 - V_2}(v_3 - v_1, v_3 - v_2) d(v_3 - v_2) \right) \right. \\ &\quad \left. \cdot \left( \int_{v_3 - v_1}^{\infty} g(\tilde{z}_1, \vartheta_3(z_3) - \vartheta_1(z_1), \vartheta_3(z_3) - \vartheta_2(z_2)) d(\vartheta_3(z_3) - \vartheta_1(z_1)) \right) \right. \\ &\quad \left. \cdot d(\vartheta_3(z_3) - \vartheta_2(z_2)) \right] d\tilde{z}_1 d(v_3 - v_1) \\ &= \int_{-\infty}^{\infty} \eta(Z_1, v_3 - v_1) d(v_3 - v_1). \end{aligned}$$

These terms can be formed from discrete choice analysis and the joint distribution of  $(Z, D_1, D_2, D_3)$ . Collecting results, we obtain

$$\begin{aligned}
\Delta_{Z_1}^{IV} &= \frac{Cov(Z_1, Y)}{Cov(Z_1, S)} \\
&= \frac{\int_{-\infty}^{\infty} \overbrace{E(Y_2 - Y_1 \mid V_2 - V_1 = v_2 - v_1, \vartheta_2(Z_2) - \vartheta_3(Z_3) \geq V_2 - V_3)}^{\text{not identified from LIV}} \overbrace{\eta(Z_1, v_2 - v_1)}^{\text{identified from discrete choice analysis and the distribution of } Z} d(v_2 - v_1)}{\int_{-\infty}^{\infty} \overbrace{E(Y_3 - Y_1 \mid V_3 - V_1 = v_3 - v_1, \vartheta_3(Z_3) - \vartheta_2(Z_2) \geq V_3 - V_2)}^{\text{not identified from LIV}} \overbrace{\eta(Z_1, v_3 - v_1)}^{\text{identified from discrete choice analysis and the distribution of } Z} d(v_3 - v_1)} \\
&= \frac{\int_{-\infty}^{\infty} \underbrace{\eta(Z_1, v_2 - v_1)}_{\text{identified from discrete choice analysis and the distribution of } Z} d(v_2 - v_1) + 2 \int_{-\infty}^{\infty} \underbrace{\eta(Z_1, v_3 - v_1)}_{\text{identified from discrete choice analysis and the distribution of } Z} d(v_3 - v_1)}{\int_{-\infty}^{\infty} \eta(Z_1, v_2 - v_1) d(v_2 - v_1) + 2 \int_{-\infty}^{\infty} \eta(Z_1, v_3 - v_1) d(v_3 - v_1)},
\end{aligned}$$

Observe that the weights do not sum to one. We cannot operationalize this decomposition even though we know the weights. What is remarkable is that we can operationalize a decomposition for the mean of  $Y_1$  against the next best alternative, which combines these terms in the fashion described in the text.

Look at what  $IV$  is doing in the general case. It is estimating flows that move from 1 to 2 and 1 to 3. The sign is opposite to the sign for the binary case. This is very peculiar.

Take the special case of the Mincer model. That setup is formulated in terms of log earnings for  $Y_1, Y_2, Y_3$ .

$$\begin{aligned}
Y_2 &= \ln(1 + g) + Y_1, \\
Y_3 &= \ln(1 + g) + Y_2,
\end{aligned}$$

where  $Y_3 = 2 \ln(1 + g) + Y_1$  and  $Y_2 = \ln(1 + g) + Y_1$ . All earnings depend on two parameters:  $(g, Y_1)$ . In this case, letting  $\alpha = \ln(1 + g)$ ,

$$\begin{aligned}
\Delta_{Z_1}^{IV} &= \frac{Cov(Z_1, Y)}{Cov(Z_1, S)} \\
&= \frac{\int_{-\infty}^{\infty} E(\alpha \mid V_2 - V_1 = v_2 - v_1, \vartheta_2(Z_2) - \vartheta_3(Z_3) \geq V_2 - V_3) \eta(Z_1, v_2 - v_1) d(v_2 - v_1)}{\int_{-\infty}^{\infty} E(\alpha \mid V_3 - V_1 = v_3 - v_1, \vartheta_3(Z_3) - \vartheta_2(Z_2) \geq V_3 - V_2) 2\eta(Z_1, v_3 - v_1) d(v_3 - v_1)} \\
&= \frac{\int_{-\infty}^{\infty} \eta(Z_1, v_2 - v_1) d(v_2 - v_1) + 2 \int_{-\infty}^{\infty} \eta(Z_1, v_3 - v_1) d(v_3 - v_1)}{\int_{-\infty}^{\infty} \eta(Z_1, v_2 - v_1) d(v_2 - v_1) + 2 \int_{-\infty}^{\infty} \eta(Z_1, v_3 - v_1) d(v_3 - v_1)},
\end{aligned}$$

so the weights on the components sum to 1. This is a special feature of the Mincer case.

$IV$  estimates the net effect of the flow from 2 to 1 and 3 to 1 but does not identify the components of the flow as does a structural model.

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