Marginal Policy Analysis and Fun with the Borel Paradox

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Potential outcomes

- Y_1 potential outcome if treated.
- Y_0 potential outcome if not treated.
- $\Delta = Y_1 - Y_0$ (Treatment Effect)
- Y observed outcome,
  $\Rightarrow Y = Y_0(1 - D) + DY_1$
  where $D = 1$ if treated, $D = 0$ otherwise.
- Implicitly fully conditioning on any observed regressors that determine Y_0 or Y_1.

Selection Model

$$D = \mathbf{1}\{P(Z) - U \geq 0\}$$

- Assumptions:
  - $P(Z)$, U, abs. continuous w.r.t. Lebesgue measure.
  - $Z \perp\!\!\!\perp (Y_0, Y_1, U)$.
- Normalization:
  - $U \sim$Unif$[0, 1]$
  - $P(Z) = \Pr[D = 1|Z]$.

Average Effect for Those on Margin

Natural parameter to consider,
Average Marginal Treatment Effect (AMTE):

$$E(Y_1 - Y_0 | P(Z) = U)$$

- Expected effect of treatment on those indifferent between treatment or not.
- Economic intuition, seems to be interesting parameter.

Paradox and "Solution"

- Borel Paradox:
  - $E(Y_1 - Y_0|P(Z) = U)$ not uniquely defined!
- Our solution:

  define $E(Y_1 - Y_0|P(Z) = U)$ by connecting to effect of marginal policy change, effect of treatment on those whose choice would be affected by marginal policy change.
- Resulting $E(Y_1 - Y_0|P(Z) = U)$ depends on particular marginal policy change (direction of marginal policy change) .

Marginal Policy Change, Effect on Those on Margin

Considering marginal policy changes and corresponding average effect for those on margin of indifference has several advantages:

- Economic, policy content.
- weak support conditions for identification
- $\sqrt{N}$-estimability.

Outline

Rest of talk:

1. Problem: Borel Paradox
2. Policy Effects
3. Marginal Policy Effects
4. Average Marginal Treatment Effects
5. Identification, Estimation Issues

Lack of Unique Definition: Borel Paradox

How to define average effect for individuals at the margin of indifference, the average marginal treatment effect?

- $D = \mathbf{1}\{P(Z) \geq U\}$.
- Suggests defining $\Delta^{AMTE} = E(Y_1 - Y_0 | P(Z) = U)$.
- Problem: not uniquely defined (Borel Paradox).
- Can by defined by
  - $E(Y_1 - Y_0 | P(Z) - U = t)$ evaluated at $t = 0$.
  - $E(Y_1 - Y_0 | P(Z)/U = t)$ evaluated at $t = 1$.
  - $E(Y_1 - Y_0 | Z\gamma - V = t)$ evaluated at $t = 0$, if $D = \mathbf{1}\{Z\gamma - V \geq 0\}$ so that $P(Z) = F_V(Z\gamma)$, $U = F_V(V)$.
  - and so forth
- Let $\Delta^{\text{MTE}}(u) \equiv E(Y_1 - Y_0 | U = u)$.

Alternative Definitions of Average Marginal Treatment Effects

$$
\begin{aligned}
E(Y_1 - Y_0 | P(Z) - U = t) &= \int_0^1 \Delta^{\text{MTE}}(u) f_P(u+t) du \\
E(Y_1 - Y_0 | Z\gamma - V = t) &= \int_0^1 \Delta^{\text{MTE}}(u) \frac{f_{Z\gamma}(F_V^{-1}(u) + t)}{E(f_V(Z\gamma - t))} du \\
E(Y_1 - Y_0 | P/U = t) &= \int_0^1 \Delta^{\text{MTE}}(u) \frac{f_P(u/t)t^{-2}u}{E(D)} du,
\end{aligned}
$$

and thus

$$
\begin{aligned}
E(Y_1 - Y_0 | P(Z) - U = 0) &= \int_0^1 \Delta^{\text{MTE}}(u) f_P(u) du \\
E(Y_1 - Y_0 | Z\gamma - V = 0) &= \int_0^1 \Delta^{\text{MTE}}(u) \frac{f_{Z\gamma}(F_V^{-1}(u))}{E(f_V(Z\gamma))} du \\
E(Y_1 - Y_0 | P/U = 1) &= \int_0^1 \Delta^{\text{MTE}}(u) \frac{f_P(u)u}{E(D)} du.
\end{aligned}
$$

Alternative Definitions of Average Marginal Treatment Effects

- $E(Y_1 - Y_0|P(Z) - U = 0) \neq E(Y_1 - Y_0|Z\gamma - V = 0)$,
  $E(Y_1 - Y_0|P(Z) - U = 0) \neq E(Y_1 - Y_0|P/U = 1)$.
- Can define $E(Y_1 - Y_0|P(Z) = U)$ using any of these limits (or in many other ways), each definition being equally valid but giving a different result.
- Thus, $E(Y_1 - Y_0|P(Z) = U)$ is not uniquely defined.
- We will define AMTE by connecting to marginal policy changes, direction of policy change will determine choice of AMTE.

Alternative Definitions of AMTE

Figure 4

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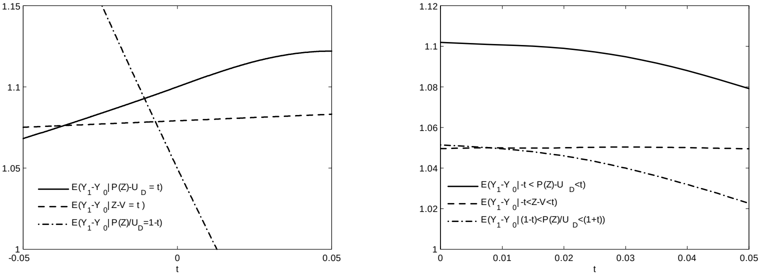


$$E\left(Y_1 - Y_0 | P(Z) - U_D = t\right) = \int_0^1 MTE(u) f_P(u+t) du;$$

$$E\left(Y_1 - Y_0 | -t < P(Z) - U_D < t\right) = \frac{\int_0^1 MTE(u)\left[F_{P(Z)}(t+u) - F_{P(Z)}(-t+u)\right] du}{\Pr(-t < P(Z) - U_D < t)}$$

$$E\left(Y_1 - Y_0 | Z - V = t\right) = \int_0^1 MTE(u) \frac{f_Z(F_V^{-1}(u_D)+t)}{E(f_V(Z-t))} du$$

PRTE

Previous Heckman-Vytlacil analysis:

- Policy Relevent Treatment Effect (PRTE): per-person effect of moving from baseline to an alternative policy, for policy alternative that affects incentives for treatment but not the potential outcomes, i.e., affect $P(Z)$ but not Y_0, Y_1.
- For example, tuition subsidies, etc.
- Effect of a discrete change, from baseline to fixed alternative policy.

PRTE

- Let $\mathcal{G}$ denote the space of all cumulative distribution functions for random variables that lie in the unit interval. Space of policy alternatives (possible distribution functions for $P(Z)$).
- Define the PRTE function, $\Delta^{\text{PRTE}} : \mathcal{G} \mapsto \mathbb{R}$, as effect of going from baseline distribution F_P of $P(Z)$ to distribution G.

PRTE

- Heckman-Vytlacil (2001) show $\Delta^{\text{PRTE}} : \mathcal{G} \mapsto \mathbb{R}$, given by

$$\Delta^{\text{PRTE}}(G) = \int_0^1 \Delta^{\text{MTE}}(u)\, \omega_{\text{PRTE}}(u; G)\, du$$

where

$$\Delta^{\text{MTE}}(u) = E(Y_1 - Y_0 | U = u),$$

$$\omega_{\text{PRTE}}(u; G) = \begin{cases} \frac{F_P(u) - G(u)}{E_G(P) - E_{F_P}(P)} & \text{if } E_G(P) \neq E_{F_P}(P) \\ 0 & \text{if } E_G(P) = E_{F_P}(P) \end{cases}$$

Effect of going from baseline to alternative policy G, as a function of G.

One Dimensional Curves in Space of Policy Alternatives

In many cases, the class of policy alternatives under consideration can be indexed by a scalar variable.

- Let P_0 denote base line probability for $D = 1$
- Let **M** denote a subset of $\mathbb{R}$ with $0 \in \mathbf{M}$,
- Let $\{P_\alpha : \alpha \in \mathbf{M}\}$ denote a class of alternative probabilities corresponding to alternative policy regimes with associated cumulative distribution functions F_{P_α}

Examples

Examples:

1. The alternative policy increases the probability of participation by $\alpha \geq 0$, so that $P_\alpha = P_0 + \alpha \Rightarrow F_{P_\alpha}(t) = F_P(t - \alpha)$.
2. The alternative policy changes each person's probability of participating by the proportion $(1 + \alpha)$, so that $P_\alpha = (1 + \alpha)P_0 \Rightarrow F_{P_\alpha}(t) = F_P(\frac{t}{1+\alpha})$.
3. Can also define alternative policy changes that operate on Z, for example, tuition subsidies or proportional tuition subsidies

Marginal PRTE

Define the Marginal Policy Relevant Treatment Effect as:

$$\Delta^{\text{MPRTE}}(F_{P_\alpha}) = \lim_{\alpha \to 0} \Delta^{\text{PRTE}}(F_{P_\alpha}),$$

Effect of a marginal change in the policy, going along the one dimensional curve $\{F_{P_\alpha}\}$. Depends on the curve $\{F_{P_\alpha}\}$, different policies will correspond to different PRTEs, and to different MPRTEs.

Marginal PRTE

Under regularity conditions,

$$\begin{aligned}\Delta^{\text{MPRTE}}(F_{P_\alpha}) &= \lim_{\alpha\to 0}\int_0^1 \Delta^{\text{MTE}}(u)\,\omega_{\text{PRTE}}(u;F_{P_\alpha})\,du \\ &= \int_0^1 \Delta^{\text{MTE}}(u)\,\omega_{\text{MPRTE}}(u;F_{P_\alpha})\,du,\end{aligned}$$

where $\omega_{\text{MPRTE}}(u;F_{P_\alpha})$ is given by

$$\omega_{\text{MPRTE}}(u;F_{P_\alpha}) = \lim_{\alpha\to 0}\left(\frac{F_P(u)-F_{P_\alpha}(u)}{E_{F_{P_\alpha}}(P)-E_{F_P}(P)}\right).$$

Examples of MPRTE

Examples of MPRTE:

1. The alternative policy increases the probability of participation by $\alpha \geq 0$, so that $P_\alpha = P_0 + \alpha \Rightarrow F_{P_\alpha}(t) = F_P(t - \alpha)$. Then

$$\begin{aligned}
\Delta^{\text{PRTE}}(F_{P_\alpha}) &= \int_0^1 \Delta^{\text{MTE}}(u) \left(\frac{F_P(u) - F_P(u - \alpha)}{\alpha} \right) du. \\
\lim_{\alpha \to 0} \Delta^{\text{PRTE}}(F_{P_\alpha}) &= \int_0^1 \Delta^{\text{MTE}}(u) f_P(u) du
\end{aligned}$$

Examples of MPRTE

2. The alternative policy changes each person's probability of participating by the proportion $(1+\alpha)$, so that $P_\alpha = (1+\alpha)P_0 \Rightarrow F_{P_\alpha}(u) = F_P(\frac{u}{1+\alpha})$. Then

$$\begin{aligned} \Delta^{\text{PRTE}}(F_{P_\alpha}) &= \int_0^1 \Delta^{\text{MTE}}(u)\left(\frac{F_P(u) - F_P(\frac{u}{1+\alpha})}{\alpha E(D)}\right) du \\ \lim_{\alpha \to 0} \Delta^{\text{PRTE}}(F_{P_\alpha}) &= \int_0^1 \Delta^{\text{MTE}}(u)\frac{u f_P(u)}{E(D)} du. \end{aligned}$$

Compared to previous example, puts higher weight on higher u values.

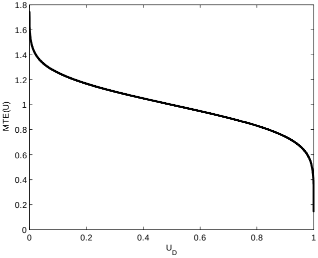
Examples of MPRTE

3. Can also develop MPRTE for alternative policy changes that operate on Z, for example, tuition subsidies or proportional tuition subsidies

Numerical Example: MTE and density of $P(Z)$

Figure 1a

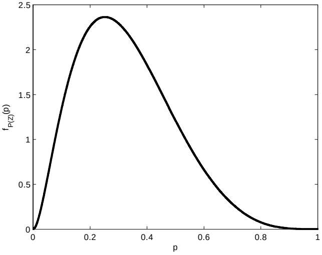
Marginal Treatment Effect



$MTE(u_D)$

Figure 1b

Density Function of $P(Z)$



$f_{P(Z)}(p)$

Numerical Example: PRTE, MPRTE for $P_\alpha = P + \alpha$

Figure 2a

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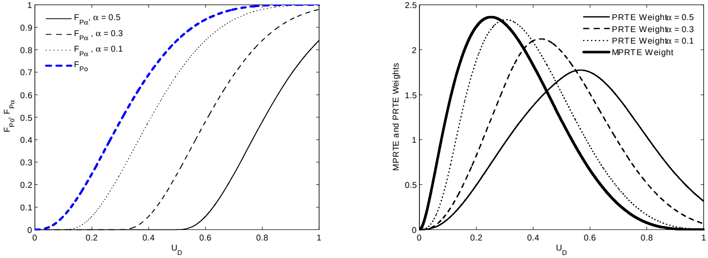


$$P_\alpha = \min\{P_0 + \alpha, 1\} \therefore F_{P_\alpha}(t) = F_P(t - \alpha)$$

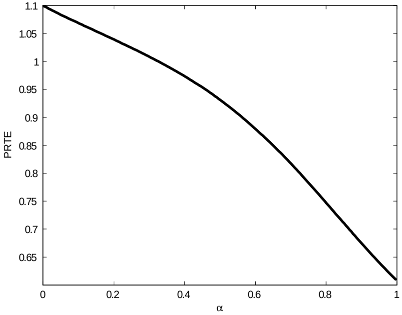
$$\Delta^{PRTE}(F_{P_\alpha}) = \int_0^1 MTE(u_D)\left(\frac{F_P(u_D) - F_P(u_D - \alpha)}{\alpha}\right) du_D$$

$$PRTE_{\alpha=0.5} = 0.93,\ PRTE_{\alpha=0.3} = 1.01,\ PRTE_{\alpha=0.1} = 1.07.$$

$$MPRTE = \lim_{\alpha\to 0} \Delta^{PRTE}(F_{P_\alpha}) = \int_0^1 MTE(u_D) f_P(u_D)\, du_D = 1.10$$

Numerical Example: PRTE for $P_\alpha = P + \alpha$, Plotted as Function of α

Figure 2b



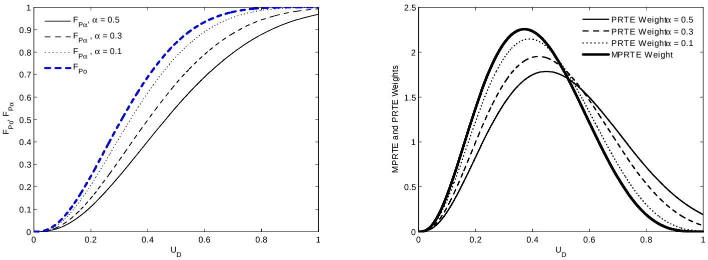
$PRTE(\alpha)$

$$P_\alpha = \min\{P_0 + \alpha, 1\} \therefore F_{P_\alpha}(t) = F_P(t - \alpha)$$

Numerical Example: PRTE, MPRTE for $P_\alpha = (1+\alpha)P$

Figure 3a

I **II**



$$P_\alpha = \min\{(1+\alpha)P_0, 1\} \therefore F_{P_\alpha}(t) = F_P\left(\frac{t}{1+\alpha}\right)$$

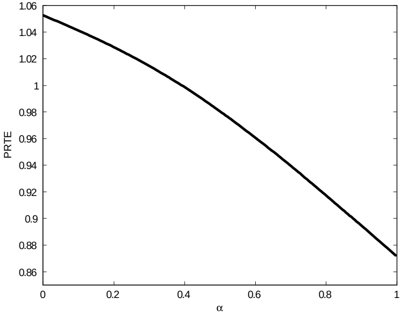
$$\Delta^{PRTE}(F_{P_\alpha}) = \int_0^1 MTE(u_D)\left(\frac{F_P(u_D) - F_P\left(\frac{u_D}{1+\alpha}\right)}{\alpha E[D]}\right) du_D$$

$$PRTE_{\alpha=0.5} = 0.98,\ PRTE_{\alpha=0.3} = 1.01,\ PRTE_{\alpha=0.1} = 1.04.$$

$$MPRTE = \lim_{\alpha\to 0} \Delta^{PRTE}(F_{P_\alpha}) = \int_0^1 MTE(u_D)\left(\frac{u \cdot f_P(u_D)}{E[D]}\right) du_D = 1.05$$

Numerical Example: PRTE for $P_\alpha = (1+\alpha)P$, Plotted as Function of α

Figure 3b



$$PRTE\,(\alpha)$$

$$P_\alpha = \min\{(1+\alpha)P_0, 1\} \therefore F_{P_\alpha}(t) = F_P\left(\frac{t}{1+\alpha}\right)$$

AMTE and Marginal PRTE

Note connections between choice of AMTE and Marginal PRTE, e.g.,

- Evaluating $E(Y_1 - Y_0|P(Z) - U = t)$ at 0 and effect of marginal policy defined by marginal additive shift in P both given by $\int_0^1 \Delta^{\text{MTE}}(u) f_P(u) du$.
- Evaluating $E(Y_1 - Y_0|P/U = t)$ at $t = 1$ and effect of marginal policy defined by marginal proportional shifts in P both given by $\int_0^1 \Delta^{\text{MTE}}(u) \frac{f_P(u)u}{E(D)} du$.
- Likewise, if $D = 1\{Z\gamma - V \geq 0\}$, then evaluating $E(Y_1 - Y_0|Z\gamma - V = t)$ at $t = 0$ leads to same expression as effect of marginal policy defined by marginal additive shift in a component of Z.

AMTE and Marginal PRTE

- Alternative definitions of AMTE correspond to alternative directions for marginal policy effects.
- Thus, uniquely define effect of treatment on those at margin of indifference by being precise about margin of indifference – e.g., those for whom a marginal additive shift versus marginal proportional shift would change treatment choice.

Identification

- Heckman-Vytlacil (2001) show all standard treatment parameters are weighted averages of MTE with weights that can be estimated.

$$\text{Treatment Parameter}\,(j) = \int_0^1 \Delta^{\text{MTE}}(u)\,\omega_j(u)\,du,$$

  where $\omega_j(u)$ is the weighting function for parameter j. We have shown same is true for AMTE/Marginal PRTE.

- $\Delta^{\text{MTE}}(u) = \frac{d}{dp}E(Y|P(Z)=p)\big|_{p=u}$
  $\Rightarrow$ identify $\Delta^{\text{MTE}}(u)$ for $u \in \text{Supp}(P(Z))$.

- $\Rightarrow$ Treatment Parameter (j) identified if $\text{Supp}(P(Z)) \supseteq \text{Supp}(\omega_j)$.

Identification

- Treatment Parameter (j) identified if $\text{Supp}(P(Z)) \supseteq \text{Supp}(\omega_j)$.
- Strong requirement for traditional treatment parameters, typically large support requirement – require 0 and/or 1 to be in support of $P(Z)$.
- But for effect of marginal policy changes, equivalently for average effect on people at margin of indifference (AMTE), support condition holds.

Estimation: Treatment Effects as Weighted Average Derivatives

Treatment Parameter (j)

$$= \int \frac{\partial}{\partial p} E(Y|P(Z)=p)\,\omega_j(p)\;dp = E(g'(P)q_j(P)),$$

where

- $g'(p) = \frac{\partial}{\partial p} E(Y|P(Z)=p)$
- $q_j(p) = \omega_j(p) / f_P(p)$

$\sqrt{N}$ – *Normal* estimation using weighted average derivative?

- Problem: $\sqrt{N}-$consistent estimability requires $q_j(p) f_P(p) = 0$ on boundary of the support of P (see, e.g., Newey and Stoker, 1993)
- Otherwise, parameter depends on conditional expectation at a point.

Estimation: Treatment Effects as Weighted Average Derivatives

- $\sqrt{N}-$consistent estimability requires $q_j(p)f_P(p) = 0$ on boundary of the support of P
- $q_j(p) = \omega_j(p)/f_P(p)$, so requires $\omega_j(p) = 0$ on boundary of the support of P
- Violated for common treatment parameters,
  - For example, $\omega_{ATE}(u) = 1$ for $u \in [0,1]$, and ATE$= E(Y|P=1) - E(Y|P=0)$.
- However, will often hold for effect of marginal policy change, equivalently, for AMTE parameter. It will often be possible to consistently estimate MPRTE and AMTE parameters at $\sqrt{N}-$rate.

Summary

- Average effect of treatment on those at margin of indifference is not uniquely defined (Borel Paradox).
- We define effect of marginal policy changes. We give unique definition to average effect on those at margin of indifference by connecting to effect of marginal policy change in a particular direction.
- Unlike traditional treatment parameters in nonparametric selection model framework, these parameters
    - Can be identified without strong support requirements.
    - Can sometimes be consistently estimated at $\sqrt{N}$-rate.