Notes on Factor Models and the Hicks Lecture Model with Normal Random Variables

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Econ 419
Winter 2008
This draft, February 22, 2008
Factor Models: Traditionally work with Covariance Information

One Factor Models

\[ E(\theta) = 0; \quad E(\epsilon_i) = 0; \quad i = 1, \ldots, 5 \]

\[ Y_1 = \alpha_1 \theta + \epsilon_1, \quad Y_2 = \alpha_2 \theta + \epsilon_2, \quad Y_3 = \alpha_3 \theta + \epsilon_3, \]
\[ Y_4 = \alpha_4 \theta + \epsilon_4, \quad Y_5 = \alpha_5 \theta + \epsilon_5, \quad \epsilon_i \perp \epsilon_j \]
For $T \geq 3$, can identify the model with on normalization.

\[
\begin{align*}
\text{Cov}(Y_1, Y_2) &= \alpha_1 \alpha_2 \sigma_\theta^2 \\
\text{Cov}(Y_1, Y_3) &= \alpha_1 \alpha_3 \sigma_\theta^2 \\
\text{Cov}(Y_2, Y_3) &= \alpha_2 \alpha_3 \sigma_\theta^2
\end{align*}
\]

Normalize $\alpha_1 = 1$

\[
\frac{\text{Cov}(Y_2, Y_3)}{\text{Cov}(Y_1, Y_2)} = \alpha_3
\]
\[ \therefore \] We know \( \sigma^2_{\theta} \) from \( \text{Cov}(Y_1, Y_2) \). From \( \text{Cov}(Y_1, Y_j), j = 3, 4, 5, \)
we know
\[ \alpha_3, \alpha_4, \alpha_5. \]

Can get the variances of the \( \varepsilon_i \) from variances of the \( Y_i \)

\[ \text{Var}(Y_i) = \alpha_i^2 \sigma^2_{\theta} + \sigma^2_{\varepsilon_i}. \]

If \( T = 2 \), all we can identify is \( \alpha_1 \alpha_2 \sigma^2_{\theta} \), even with the normalization.

If \( \alpha_1 = 1, \sigma^2_{\theta} = 1 \), we identify \( \alpha_2 \).
2 Factors:

Assume $\theta_1 \perp \perp \theta_2$

$\varepsilon_i \perp \perp \varepsilon_j \ \forall i, j$

Normalize:

\[
Y_1 = \alpha_{11} \theta_1 + (0) \theta_2 + \varepsilon_1 \\
Y_2 = \alpha_{21} \theta_1 + (0) \theta_2 + \varepsilon_2 \\
Y_3 = \alpha_{31} \theta_1 + \alpha_{32} \theta_2 + \varepsilon_3 \\
Y_4 = \alpha_{41} \theta_1 + \alpha_{42} \theta_2 + \varepsilon_4 \\
Y_5 = \alpha_{51} \theta_1 + \alpha_{52} \theta_2 + \varepsilon_5
\]

Let $\alpha_{11} = 1, \alpha_{32} = 1$. 
\[ \text{Cov} (Y_1, Y_2) = \alpha_{21} \sigma^2_{\theta_1} \]
\[ \text{Cov} (Y_1, Y_3) = \alpha_{31} \sigma^2_{\theta_1} \]
\[ \text{Cov} (Y_2, Y_3) = \alpha_{21} \alpha_{31} \sigma^2_{\theta_1} \]

Form ratio of \[ \frac{\text{Cov} (Y_2, Y_3)}{\text{Cov} (Y_1, Y_2)} = \alpha_{31}, \]

\[ \therefore \text{we identify } \alpha_{31}, \alpha_{21}, \sigma^2_{\theta_1}, \text{ as before.} \]
\[ \text{Cov} (Y_1, Y_4) = \alpha_{41} \sigma^2_{\theta_1} , \]
\[ \vdots \]
\[ \text{Cov} (Y_1, Y_k) = \alpha_{k1} \sigma^2_{\theta_1} \]

\[ \therefore \text{we identify } \alpha_{k1} \text{ for all } k \text{ and } \sigma^2_{\theta_1}. \]
\[ \text{Cov}(Y_3, Y_4) - \alpha_{31} \alpha_{41} \sigma_{\theta_1}^2 = \alpha_{42} \sigma_{\theta_2}^2 \]
\[ \text{Cov}(Y_3, Y_5) - \alpha_{31} \alpha_{51} \sigma_{\theta_1}^2 = \alpha_{52} \sigma_{\theta_2}^2 \]
\[ \text{Cov}(Y_4, Y_5) - \alpha_{41} \alpha_{51} \sigma_{\theta_1}^2 = \alpha_{52} \alpha_{42} \sigma_{\theta_2}^2 \]

By same logic,

\[ \frac{\text{Cov}(Y_4, Y_5) - \alpha_{41} \alpha_{51} \sigma_{\theta_1}^2}{\text{Cov}(Y_3, Y_4) - \alpha_{31} \alpha_{41} \sigma_{\theta_1}^2} = \alpha_{52} \]

\[ \therefore \text{get } \sigma_{\theta_2}^2 \text{ and the factor “2” loadings.} \]
If we have dedicated measurements of factor, do not need a normalization on $Y$. They provide a natural scale. Assume $\theta_1 \perp \perp \theta_2$ (testable)

$$M_1 = \theta_1 + \varepsilon_{1M}$$
$$M_2 = \theta_2 + \varepsilon_{2M}$$

\[
\begin{align*}
\text{Cov}(Y_1, M_1) &= \alpha_{11}\sigma_{\theta_1}^2 \\
\text{Cov}(Y_2, M_1) &= \alpha_{21}\sigma_{\theta_1}^2 \\
\text{Cov}(Y_3, M_1) &= \alpha_{31}\sigma_{\theta_1}^2 \\
\text{Cov}(Y_1, Y_2) &= \alpha_{11}\alpha_{21}\sigma_{\theta_1}^2, \\
\text{Cov}(Y_1, Y_3) &= \alpha_{11}\alpha_{31}\sigma_{\theta_1}^2, \quad \therefore \alpha_{21}\sigma_{\theta_1}^2.
\end{align*}
\]

\[\therefore\text{ We can get } \alpha_{21}, \sigma_{\theta_1}^2 \text{ and the other factors.}\]
General Case

\[ Y_{T \times 1} = \mu + \Lambda_{T \times K \times 1} \theta + \varepsilon_{T \times 1} \]

\( \theta \) are factors, \( \varepsilon \) uniquenesses

\[ E(\varepsilon) = 0 \]

\[ \text{Var}(\varepsilon \varepsilon') = D = \begin{pmatrix} \sigma^2_{\varepsilon_1} & 0 & \cdots & 0 \\ 0 & \sigma^2_{\varepsilon_2} & 0 & : \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & \sigma^2_{\varepsilon_T} \end{pmatrix} \]

\[ E(\theta) = 0 \]

\[ \text{Var}(Y) = \Lambda \Sigma_\theta \Lambda' + D \quad \Sigma_\theta = E(\theta \theta') \]
The only source of information on $\Lambda$ and $\Sigma_\theta$ is from the covariances.

Associated with each variance of $Y_i$ is a $\sigma^2_{\xi_i}$.

Each variance contributes one new parameter.

How many unique covariance terms do we have?

$$\frac{T(T-1)}{2}$$ This is the data.

We have $T$ uniquenesses; $TK$ elements of $\Lambda$. 
\[ K \left( \frac{K - 1}{2} \right) \] elements of \( \Sigma_\theta \).

\[ K \left( \frac{K - 1}{2} \right) + TK \] parameters \((\Sigma_\theta, \Lambda)\).

Observe that if we multiply \( \Lambda \) by an orthogonal matrix \( C \), \((CC' = I)\), we have

\[
\text{Var}\,(Y) = \Lambda C' [\Sigma_\theta C] C' \Lambda' + D
\]

\( C \) is a “rotation”. Cannot separate \( \Lambda C \) from \( \Lambda \).

Model not identified against orthogonal transformations in the general case.
Some common assumptions:

(i) $\theta_i \perp \perp \theta_j, \forall i \neq j$

$$
\Sigma_\theta = \begin{pmatrix}
\sigma^2_{\theta_1} & 0 & \cdots & 0 \\
0 & \sigma^2_{\theta_2} & 0 & \vdots \\
\vdots & 0 & \ddots & \vdots \\
0 & \cdots & 0 & \sigma^2_{\theta_K}
\end{pmatrix}
$$
joined with

(ii)

\[ \Lambda = \begin{pmatrix}
1 & 0 & 0 & 0 & \cdots & 0 \\
\alpha_{21} & 0 & 0 & 0 & \cdots & 0 \\
\alpha_{31} & 1 & 0 & 0 & \cdots & 0 \\
\alpha_{41} & \alpha_{42} & 0 & 0 & \cdots & 0 \\
\alpha_{51} & \alpha_{52} & 1 & 0 & \cdots & 0 \\
\alpha_{61} & \alpha_{62} & \alpha_{63} & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & 1 \\
\end{pmatrix} \]
We know that we can identify of the $\Lambda, \Sigma_{\theta}$ parameters.

\[
\frac{K(K-1)}{2} + TK \leq \frac{T(T-1)}{2}
\]

“Ledermann Bound”
Generalized Roy Model with Factor Structure

Generalized Roy versions of college choice model:

\[ M = \mu(X) + \theta_1 \alpha_{1,M} + \theta_2 \alpha_{2,M} + \varepsilon_M \]

(Measurement: A test score equation)

\[
\begin{align*}
Y_1^1 &= \mu_1^1(X) + \theta_1 \alpha_{1,1}^1 + \theta_2 \alpha_{2,1}^1 + \varepsilon_1^1 \\
Y_2^1 &= \mu_2^1(X) + \theta_1 \alpha_{1,2}^1 + \theta_2 \alpha_{2,2}^1 + \varepsilon_2^1 \\
\end{align*}
\]

\(\text{College earnings}\)

\[
\begin{align*}
Y_1^0 &= \mu_1^0(X) + \theta_1 \alpha_{1,1}^0 + \theta_2 \alpha_{2,1}^0 + \varepsilon_1^0 \\
Y_2^0 &= \mu_2^0(X) + \theta_1 \alpha_{1,2}^0 + \theta_2 \alpha_{2,2}^0 + \varepsilon_2^0 \\
\end{align*}
\]

\(\text{High School earnings}\)

Cost

\[ C = Z \gamma + \theta_1 \alpha_{1C} + \theta_2 \alpha_{2C} + \varepsilon_C \]
Decision Rule Under Perfect Certainty:
(Assume Interest Rate $r = 0$)

\[ I = \mu_1^1(X) + \mu_2^1(X) + \theta_1 (\alpha^1_{1,1} + \alpha^1_{1,2}) \\
+ \theta_2 (\alpha^1_{2,1} + \alpha^1_{2,2}) + \varepsilon^1_1 + \varepsilon^1_2 \\
- \left[ \mu_1^0(X) + \mu_2^0(X) + \theta_1 (\alpha^0_{1,1} + \alpha^0_{1,2}) \\
+ \theta_2 (\alpha^0_{2,1} + \alpha^0_{2,2}) + \varepsilon^0_1 + \varepsilon^0_2 \right] \\
- Z\gamma - \theta_1 \alpha_{1C} - \theta_2 \alpha_{2C} - \varepsilon_C \\
= \mu_1^1(X) + \mu_2^1(X) - \left[ \mu_1^0(X) + \mu_2^0(X) + Z\gamma \right] \\
+ \theta_1 \left[ (\alpha^1_{1,1} + \alpha^1_{1,2}) - (\alpha^0_{1,1} + \alpha^0_{1,2}) - \alpha_{1C} \right] \\
+ \theta_2 \left[ (\alpha^1_{2,1} + \alpha^1_{2,2}) - (\alpha^0_{2,1} + \alpha^0_{2,2}) - \alpha_{2C} \right] \\
+ (\varepsilon^1_1 + \varepsilon^1_2) - (\varepsilon^0_1 + \varepsilon^0_2) - \varepsilon_C \]
In Reduced Form

\[ I = \varphi(X, Z) + \alpha_{I,1}\theta_1 + \alpha_{I,2}\theta_2 + \varepsilon_I. \]

Set \( U_I = \alpha_{I,1}\theta_1 + \alpha_{I,2}\theta_2 + \varepsilon_I. \)

\[ \therefore \text{we can write} \]

\[ Y_{11} = \mu_{11}(X) + U_{11} \]
\[ Y_{12} = \mu_{12}(X) + U_{12} \]
\[ Y_{01} = \mu_{01}(X) + U_{01} \]
\[ Y_{02} = \mu_{02}(X) + U_{02} \]

\( U_{11}, U_{12} \) etc. match the error terms previously shown.

\[ U_{11} = \theta_1\alpha_{1,1} + \theta_2\alpha_{2,1} + \varepsilon_1 \text{ etc.} \]
\[ U_M = \theta_1\alpha_{1,M} + \theta_2\alpha_{2,M} + \varepsilon_M \]
\[
E \left( Y_1^1 \mid X, Z, I > 0 \right) = \mu_1^1 (X) + \frac{Cov (U_1^1, I)}{Var (I)} \lambda (I)
\]

Using notes on the Roy model, we can identify beside the means,

\[
\mu_1^1 (X), \mu_2^1 (X), \mu_0^2 (X), \mu_2^0 (X), \mu_2^0 (X),
\]

the following parameters:

\[
Cov (U_1^1, U_2^1), Var (U_1^1), Var (U_2^1),
Cov (U_1^1, U_M^1), Cov (U_2^1, U_M^1), Var (U_M^1),
Cov (U_1^0, U_2^0), Var (U_1^0), Var (U_2^0),
Cov (U_1^0, U_M^0), Cov (U_2^0, U_M^0)
\]
Normal Case: \((\theta, \varepsilon)\) normal.

\[(\theta, \varepsilon) \perp \perp (X, Z)\]

\[
\Pr (S = 1 \mid X, Z, \theta_1, \theta_2) = \Phi \left[ \frac{1}{\sigma_{\varepsilon_1}} \left[ \mu_1^1(X) + \mu_2^1(X) - [\mu_1^0(X) + \mu_2^0(X)] - [Z \gamma + \theta_1 \alpha_{I,1} + \theta_2 \alpha_{I,2}] \right] \right]
\]
Fact:

If $S = I[X\beta + \theta > V]$, $X \perp \perp (\theta, V)$, $\theta, V$ are normal, $\theta \perp \perp V$, $E(\theta) = 0, E(V) = 0$

$$\Pr(S = 1 \mid X, \theta) = \Phi \left( \frac{X\beta + \theta}{\sigma_V} \right)$$

$$\Pr(S = 1 \mid X) = \Phi \left( \frac{X\beta}{(\sigma_V^2 + \sigma_\theta^2)^{\frac{1}{2}}} \right)$$

Why?

$S = I[X\beta > V - \theta]$.

Rest follows from independence (between $V - \theta$, and $X$, and normality).
Unconditional Probability: (Not conditional on Factors)

\[
\Pr (S = 1 \mid X, Z) = \Phi \left[ \frac{\mu_1^1 (X) + \mu_2^1 (X) - [\mu_1^0 (X) + \mu_2^0 (X)] - Z \gamma}{\left( \sigma_{\varepsilon_1}^2 + \alpha_{\theta_1}^2 \sigma_{\beta_1}^2 + \alpha_{\theta_2}^2 \sigma_{\beta_2}^2 \right)^{1/2}} \right]
\]

Observe that if we know \( \mu_1^1 (X), \mu_1^2 (X), \mu_0^1 (X), \mu_0^2 (X) \) we know

\[
[\mu_1^1 (X) + \mu_2^1 (X)] - [\mu_1^0 (X) + \mu_2^0 (X)].
\]

If \( Z \gamma \) not perfectly collinear with this term (e.g. one \( X \) or more not in \( Z \)) we can identify

\[
\left( \sigma_{\varepsilon_1}^2 + \alpha_{\theta_1}^2 \sigma_{\beta_1}^2 + \alpha_{\theta_2}^2 \sigma_{\beta_2}^2 \right)^{1/2}
\]

\( \therefore \) we also identify \( \gamma \) (get absolute scale on costs).
Suppose agents do not know $\theta_2$ or the future $\varepsilon_1, \varepsilon_2, \varepsilon_1^0, \varepsilon_2^0$ but know $\varepsilon_c$ and $\theta_1$.

Then if what they know is set at mean zero, (they use rational expectations in a linear decision rule) and their mean forecast is the population mean,

$$\sigma_{\varepsilon_1}^2 = \sigma_{\varepsilon_c}^2$$

and $\alpha_{1,2} = 0$, what can we identify?
What information do we have about covariances?

Suppose we have two dedicated measurement systems for $\theta_1$ and $\theta_2$. We normalize the First loading as a convention.

$$
\begin{align*}
M_1^1 &= \theta_1 + \varepsilon_{1,M}^1 \\
M_2^1 &= \alpha_{2,M}^1 \theta_1 + \varepsilon_{2,M}^1 \\
M_3^1 &= \alpha_{3,M}^1 \theta_1 + \varepsilon_{3,M}^1 \\
M_1^2 &= \theta_2 + \varepsilon_{1,M}^2 \\
M_2^2 &= \alpha_{2,M}^2 \theta_2 + \varepsilon_{2,M}^2 \\
M_3^2 &= \alpha_{3,M}^2 \theta_2 + \varepsilon_{3,M}^2
\end{align*}
\right\}
\begin{align*}
\text{Cognitive Ability} \\
\text{Noncognitive Ability}
\end{align*}
$$
Observe from $M^1$ system we get

$$Var(\theta_1), \alpha_{2,M}^1, \alpha_{3,M}^1$$

From $M^2$ system we get

$$Var(\theta_2), \alpha_{2,M}^2, \alpha_{3,M}^2$$
Then
\[
\begin{align*}
\text{Cov} (U_1^1, M_1^1) &= \alpha_{1,1}^1 \sigma^2_{\theta_i} \\
\text{Cov} (U_2^1, M_1^1) &= \alpha_{1,2}^1 \sigma^2_{\theta_i}
\end{align*}
\]
∴ we get all of the factor loadings in \(Y^1\) on \(\theta_1\).

Using \(M_1^2\) we get \(\alpha_{2,1}^1, \alpha_{2,2}^1\) and we get variances of uniquenesses \(\text{Var} (\varepsilon_1^1), \text{Var} (\varepsilon_2^1)\).

By similar reasoning, we get
\[
\begin{align*}
\alpha_{1,1}^0, \alpha_{2,1}^0, \alpha_{1,2}^0, \alpha_{2,2}^0 \\
\text{Var} (\varepsilon_1^0), \text{Var} (\varepsilon_2^0)
\end{align*}
\]
Observe that from
\[
\text{Cov} (I, M_1^1) = \sigma_1^2 \left[ \alpha_{1,1}^1 + \alpha_{1,2}^1 - (\alpha_{1,1}^0 + \alpha_{1,2}^0) - \alpha_{1,C} \right]
\]
∴ We can get \( \alpha_{1C} \), since we know all other terms on the right hand side by the previous reasoning.

From
\[
\text{Cov} (I, M_1^2) = \sigma_2^2 \left[ \alpha_{2,1}^1 + \alpha_{2,2}^1 - (\alpha_{2,1}^0 + \alpha_{2,2}^0) - \alpha_{2,C} \right]
\]
we can get \( \alpha_{2C} \).

From \( \text{Pr} (S = 1 \mid X, Z) \), we can identify \( \sigma_{\varepsilon_j}^2 \) using previous reasoning.
Therefore we can identify everything in the model if there is one $X$ not in $Z$ since we can identify the terms in the numerator.
Can we test the model?

In the notation of the Hicks lecture notes, we have for a test of whether $\theta_2$ belongs in the model

$$
\Pr (S = 1 \mid X, Z) = \Phi \left[ \frac{\mu_1^1 (X) + \mu_2^1 (X) - [\mu_1^0 (X) + \mu_2^0 (X)] - Z \gamma}{\left( \sigma_{\varepsilon t}^2 + \alpha_{I,1}^2 \sigma_{\theta_1}^2 + \alpha_{I,2}^2 \sigma_{\theta_2}^2 \Delta \theta_2 \right)^{\frac{1}{2}}} \right]
$$

Apparently, we can test the null

$$H_0 : \Delta \theta_2 = 0$$

\therefore, we can test if $\theta_2$ components enter or not.
The problem with this test is that if $\sigma^2_{\varepsilon_c} \neq 0$, we can always adjust its value to fit the model perfectly well.
(This problem vanishes if we assume a pure Roy model (so $\sigma^2_{\varepsilon_c} = 0$).)

Notice, however, that we can also tolerate $\gamma \neq 0$ so long as $\sigma^2_{\varepsilon_c} = 0$. 
Correct idea of the correct test:

Form

\[ \text{Cov} \left( \frac{I}{\sigma_1}, U_1^1 \right) = \frac{\sigma^2_{\theta_2} \alpha_{1,1}^1}{\sigma_1} \left[ \alpha_{1,1}^1 + \alpha_{1,2}^1 - (\alpha_{1,1}^0 + \alpha_{1,2}^0) - \alpha_{1,c} \right] \]
\[ + \Delta_{\theta_2} \sigma^2_{\theta_2} \alpha_{1,2}^1 \left[ \alpha_{1,1}^1 + \alpha_{1,2}^1 - (\alpha_{1,1}^0 + \alpha_{1,2}^0) - \alpha_{1,c} \right] \]

\[ \therefore \text{we can compute the test under the null.} \]

Under the null that \( \Delta_{\theta_2} = 0 \), we can identify \( \sigma^2_{\varepsilon_c} \).

\[ \therefore \text{we construct a test under null:} \]

\[ \text{Cov} \left( \frac{I}{\sigma_1}, U_1^1 \right) = \frac{\sigma^2_{\theta_2} \alpha_{1,1}^1}{\sigma_1} \left[ \alpha_{1,1}^1 + \alpha_{1,2}^1 - (\alpha_{1,1}^0 + \alpha_{1,2}^0) - \alpha_{1,c} \right] = 0 \]

We know both terms under the null. Departures are evidence that agents know \( \theta_2 \).
If the agent knows $\theta_1$ but not $\theta_2$ and sets

$$E(\theta_2) = 0.$$ 

Justified by linearity of the criterion and rational expectations, assuming $E(\theta_2 \mid I_0) = 0$. 
Then we have that the test amounts to deciding

- Which model fits the data better?

Average effect (we estimate the average probability):

\[
\int \Pr (S = 1 \mid X, Z, \theta_1, \Delta \theta_2, \theta_2) f (\theta_1) f (\theta_2) d\theta.
\]

(we test \(\Delta \theta_2 = 0\))

This is what is done in the Hicks lecture.