Notes on Frisch Demands*

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Simple Two Good Model to Interpret Frisch Demands

This is a model for each period of a general life cycle model. Abstract from interest rates and time preference.

$$\max U(C, L)$$

s.t. $$PC + WL = E$$

$E$ is the resources available to be spent within the period. Assume fixed for the moment.

$$U_C = \lambda P$$

$$U_L = \lambda W$$

$C = \text{Consumption}$

$L = \text{Leisure}$

$\lambda = \text{LaGrange Multiplier associated with the budget constraint}$

Treat leisure as an ordinary good for the moment.
Comparative Statics of Model

Totally Differentiate

\[
\begin{bmatrix}
U_{CC} & U_{CL} & -P \\
U_{CL} & U_{LL} & -W \\
-P & -W & 0
\end{bmatrix}
\begin{bmatrix}
dC \\
dL \\
d\lambda
\end{bmatrix}
= 
\begin{bmatrix}
\lambda dP \\
\lambda dW \\
-dE + CdP + LdW
\end{bmatrix}
\]

Let bars denote determinants

\[
\frac{\partial L}{\partial E} = \left| \begin{array}{ccc}
U_{CC} & 0 & -P \\
U_{CL} & 0 & -W \\
-P & -1 & 0
\end{array} \right|
\]

Notation \( | \cdot | = \left| \begin{array}{ccc}
U_{CC} & U_{CL} & -P \\
U_{CL} & U_{LL} & -W \\
-P & -W & 0
\end{array} \right| > 0 \) (second order conditions)
\[
\frac{dL}{dE} = \begin{cases} 
\left( -U_{CC}W + PU_{CL} \right) \cdot \left| \begin{array}{ccc}
>0 \\
\end{array} \right. 
\end{cases} > 0 \text{ (From Assumed Normality)}
\]

Now this can be related to displacements for \( \lambda \):

\[
\frac{\partial \lambda}{\partial W} \bigg|_{E=\bar{E}} = \begin{vmatrix}
U_{CC} & U_{CL} & 0 \\
U_{CL} & U_{LL} & \lambda \\
-P & -W & 0 \\
\end{vmatrix} \cdot 
\]

\[
= -\lambda \begin{vmatrix}
U_{CC} & U_{CL} \\
-P & -W \\
\end{vmatrix} 
\]

\[
= (-\lambda) \left( -U_{CC}W + U_{CL}P \right) \cdot \left| \begin{array}{ccc}
>0 \\
\end{array} \right. 
\]

\[
= \lambda \left( U_{CC}W - U_{CL}P \right) \cdot \left| \begin{array}{ccc}
>0 \\
\end{array} \right. 
\]

\(< 0 \text{ from normality of leisure} \)

Thus

\[
\frac{\partial \lambda}{\partial W} \bigg|_{E=\bar{E}} = -\frac{\partial L}{\partial E}
\]

(Intuition: The cost of a component of utility rises, the marginal utility of income declines.)
Observe

\[
\frac{\partial \lambda}{\partial E} = - \left| \begin{array}{cc}
U_{CC} & U_{CL} \\
U_{CL} & U_{LL}
\end{array} \right| 
\]

\[
d\lambda < 0 \text{ under strict concavity}
\]

(Diminishing Marginal Utility of Income)

Take total differential:

\[
d\lambda = \left. \frac{\partial \lambda}{\partial P} \right|_{E=E} dP + \left. \frac{\partial \lambda}{\partial W} \right|_{E=E} dW + \left. \frac{\partial \lambda}{\partial E} \right|_{E=E} (dE - CdP - LdW)
\]

(This sign pattern assumes that all goods are normal.)
Consider
\[ \frac{\partial L}{\partial W} \bigg|_\lambda \]

Compensate the agent to keep \( \lambda \) fixed. Let \( dK \) be the compensation.

\[ d\lambda = 0 = \left( \frac{\partial \lambda}{\partial W} \right)_{\text{U}} \, dW + \left( \frac{\partial \lambda}{\partial E} \right) (dK - LdW) \]

\[ 0 = \frac{\partial \lambda}{\partial E} dW - LdW + dK \]

\[ dK = \left( L - \frac{\partial \lambda}{\partial W} \bigg|_{\text{U}} \right) dW \]

\[ \therefore \text{The compensation required to keep } \lambda \text{ constant is smaller than what is required to keep utility constant for the same change in the wage.} \]
So

Hicks Slutsky compensated substitution effect

\[
\left. \frac{\partial L}{\partial W} \right|_{\lambda} = \left( \left. \frac{\partial L}{\partial W} \right|_{U=\overline{U}}^{(-)} - L \left. \frac{\partial L}{\partial E} \right|_{U=\overline{U}}^{(+)} \right) + \left. \frac{\partial L}{\partial E} \right|_{U=\overline{U}} \left( \frac{\partial K}{\partial W} \right)
\]

compensation to keep \( \lambda \) fixed

\[
\left. \frac{\partial L}{\partial W} \right|_{\lambda} = \left. \frac{\partial L}{\partial W} \right|_{U=\overline{U}}^{(-)} + \left. \frac{\partial L}{\partial E} \right|_{U=\overline{U}}^{(+)} \left( \frac{\partial K}{\partial W} - L \right)
\]

\[
= \left. \frac{\partial L}{\partial W} \right|_{U=\overline{U}}^{(-)} + \left. \frac{\partial L}{\partial E} \right|_{U=\overline{U}}^{(+)} \left( - \frac{\partial \lambda}{\partial W} \right)
\]

Clearly \( \left. \frac{\partial L}{\partial W} \right|_{\lambda} \leq \left. \frac{\partial L}{\partial W} \right|_{U=\overline{U}} \leq 0 \)
Digression

Direct derivation (assume $\lambda$ fixed).

$$U_C = \lambda P$$
$$U_L = \lambda W$$

$$U_{CC} dC + U_{CL} dL = \lambda dP$$
$$U_{CL} dC + U_{LL} dL = \lambda dW$$
Take Cramer’s Rule:

\[
 \frac{\partial L}{\partial W} \bigg|_\lambda = \begin{vmatrix}
 U_{CC} & 0 \\
 U_{CL} & \lambda \\
 U_{CC} & U_{CL} \\
 U_{CL} & U_{LL}
\end{vmatrix}
\]

\[
= \frac{U_{CC} \lambda}{U_{CC} U_{LL} - U_{CL}^2} < 0
\]

Claimed result follows because

\[
U_{CC} < 0
\]

\[
U_{CC} U_{LL} - U_{CL}^2 > 0
\]

(Concavity)
\[
\left. \frac{\partial C}{\partial W} \right|_\lambda = \frac{\begin{vmatrix} 0 & U_{CL} \\ \lambda & U_{LL} \end{vmatrix}}{\begin{vmatrix} U_{CC} & U_{CL} \\ U_{CL} & U_{LL} \end{vmatrix}} = \frac{-\lambda U_{CL}}{U_{CC} U_{LL} - U_{CL}^2}
\]

\[U_{CL} > 0 \implies \left. \frac{\partial C}{\partial W} \right|_\lambda < 0\]

\[U_{CL} < 0 \implies \left. \frac{\partial C}{\partial W} \right|_\lambda > 0\]

(See e.g. Heckman, 1974, AER)
Back to the main thread:

\[ 0 \geq \frac{\partial L}{\partial W} \bigg|_{u=\bar{u}} \geq \frac{\partial L}{\partial W} \bigg|_{\lambda} \]

Intuition: we have to pay less compensation to keep \( \lambda \) fixed than \( U \) fixed and leisure is a normal good, so we get less leisure with \( \lambda \) fixed and hence the \( \lambda \)-constant wage response is more negative.

\[ \frac{\partial L}{\partial W} \bigg|_{u=\bar{u}} \geq \frac{\partial L}{\partial W} \] (From normality of leisure)

Thus we do not know how to order

\[ \frac{\partial L}{\partial W} \text{ and } \frac{\partial L}{\partial W} \bigg|_{\lambda} \]

uncompensated
We have thus far abstracted from the fact that people have a time endowment, $T$. (Implicitly, we set $T = 0$). The Hicks-Slutsky uncompensated wage effect for leisure is (for $T \geq 0$)

\[
\frac{\partial L}{\partial W} = \frac{\partial L}{\partial W} \bigg|_{U = \bar{U}} + (T - L)\frac{\partial L}{\partial E}
\]

This makes the wage effect on labor supply ambiguous.

Define $h = T - L$

\[
\frac{\partial h}{\partial L} = -1
\]

\[
\frac{\partial h}{\partial W} = \frac{\partial h}{\partial W} \bigg|_{U = \bar{U}} + h \left( \frac{\partial h}{\partial E} \right)
\]

\[
\frac{\partial h}{\partial W} \bigg|_{\lambda} = \frac{\partial h}{\partial W} \bigg|_{U = \bar{U}} + \frac{\partial h}{\partial E} h + \frac{\partial h}{\partial E} \left( \frac{\partial K}{\partial W} \right)
\]
What is $dK$?

\[ d\lambda = 0 = \left( \frac{\partial \lambda}{\partial W} \right) \bigg|_U dW + \left( \frac{\partial \lambda}{\partial E} \right) (dK + hdW) \]

\[ 0 = \frac{\partial \lambda}{\partial W} \bigg|_U dW + hdW + dK \]

\[ \frac{\partial K}{\partial W} = - \left( \frac{\partial \lambda}{\partial W} \bigg|_U + h \right) < 0 \]

\[ \underbrace{+} \]
So

\[
\frac{\partial h}{\partial W} \bigg|_\lambda = \frac{\partial h}{\partial W} \bigg|_{U=\bar{U}} \frac{\partial h}{\partial W} \bigg|_{U=\bar{U}} \left( \frac{\partial \lambda}{\partial E} \bigg|_{U=\bar{U}} \right) + \left( \frac{\partial \lambda}{\partial E} \bigg|_{U=\bar{U}} \right)
\]

\[
\therefore \ 0 \leq \frac{\partial h}{\partial W} \bigg|_{U=\bar{U}} \leq \frac{\partial h}{\partial W} \bigg|_{U=\bar{U}} \lambda
\]

and with respect to the uncompensated Marshallian elasticity:

\[
\frac{\partial h}{\partial W} < \frac{\partial h}{\partial W} \bigg|_{U=\bar{U}} \leq \frac{\partial h}{\partial W} \bigg|_{U=\bar{U}} \lambda
\]

In the case of labor supply more income is transferred out periods where wages increase. \(dK\) is larger (more negative) in this case than the previous one considered.

(There is a more negative wage effect on borrowing, \(i.e.\) a force toward saving. This is intuitively so because wage increase generates income in the period which is likely transferred to other periods.)
This is for a single period.

We connect back to the multiperiod model by using the intertemporal arbitrage condition

\[ \lambda_{t+1} = \frac{1 + \beta}{1 + r} \lambda_t \]

(see BHH).

This determines the allocations of the \( E_t \) across the periods in a two stage budgeting procedure.

In those periods where \( \lambda_t \) is high ship resources in (borrow).

Where it is low, ship out (save).

As we showed, holding \( U = \bar{U} \), \( W \uparrow \lambda \downarrow \).

Tends to create a force for saving in periods of wage increases.

Possibly offsetting it, is substitution toward consumption when leisure goes down.