Tinbergen and Rosen

Hedonic Models

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Why is the Study of Hedonic Markets Important?

- Quality and variety are central features of a modern economy.

(1) As wealth rises, technology improves. It is quality of goods per capita that increases, not mainly their quantity. The same is true of labor force. As scale increases, so does the possibility for comparative advantage and investment in specialized skills which are a source of improvement for modern economies.
Why is the Study of Hedonic Markets Important?

(2) Traditional approach in economics in the mid 1950’s assumes uniform quality of a finite number of goods. Tinbergen and Theil in mid 1950’s in Rotterdam were among the first to recognize the importance of heterogeneity, variety and quality in economics and how to model it.
Why is the Study of Hedonic Markets Important?

- Economics of quality developed to date has a substantial impact on our understanding of:

  1. Consumer Price Indices
  2. Wage Inequality
  3. How Regulation and Deregulation Affect Consumer Welfare
Why is the Study of Hedonic Markets Important?

Questions:

(1) How to formulate the problem in more general cases?

(2) How to make it empirically operational, testable on data and useful for policy analysis?

Tinbergen made a fundamental but not yet fully appreciated contribution to the study of hedonics.
Acknowledgments

- This presentation draws on joint work with:

  1. Ivar Ekeland (UBC)
  2. Rosa Matzkin (Northwestern University)
  3. Lars Nesheim (UCL)
Introduction to Hedonic Markets

- Study competitive hedonic markets, markets for heterogeneous goods in which characteristics of goods are priced out.

- Rosen proposed an identification strategy to recover preferences and technology.

- Widely believed that parameters of hedonic models are identified only through arbitrary functional form and exclusion assumptions, especially when they are estimated from data on a single market.
Introduction to Hedonic Markets

Main Conclusion to be Established

- Hedonic model is generically nonlinear.

- This nonlinearity is a source of identification, even in single markets.

- Linearity is an arbitrary and misleading functional form in the context of identifying and estimating hedonic models.
Introduction to Hedonic Markets

- The economic model for which widely used linearization methods are exact is implausible.

- Commonly used linearization strategies produce identification problems.
Introduction to Hedonic Markets

- In a wide class of additive, parametric models, parameters are generically identified with data from a single market.

- In additive nonparametric models, parameters are identified up to affine transformations.

- In general nonlinear models, identified up to monotone transformations.
Introduction to Hedonic Markets

- Identification analysis also applies to other nonlinear pricing models.

  (1) Effects of taxes on behavior when taxes are set optimally (Mirrlees (1971))

  (2) Monopoly pricing (Mussa and Rosen (1978))

  (3) Taxes and labor supply (Heckman (1974); Hausman (1980))

  (4) Social interactions and sorting (Nesheim (2001))

- Analysis can be extended to non-additive models (Heckman, Matzkin, and Nesheim (2002)).
Outline of Lecture

- Section 3 presents the hedonic model and section 4 reviews an important quadratic case due to Tinbergen (1956), and used by Epple (1987).

(1) Equilibrium price depends on production technology parameters, consumer preference parameters, and the distribution of heterogeneity in the population.

(2) Tinbergen model has closed form solution resulting in quadratic pricing function.
Outline of Lecture: Section 4

- Section 4 focuses on a quadratic model to motivate discussion of economic and empirical properties of hedonic models.

- S. Rosen’s empirical strategy.
Outline of Lecture: Section 4

Brown and H. Rosen’s (1982) critique:

(1) Identification depends on arbitrary functional form restrictions.

(2) $z$ is endogenous; no instruments are available.

(3) Multimarket data can help but caution is required.
Outline of Lecture: Section 4

- Critique is not valid when quadratic model is perturbed.
  
  (1) Non-identification is *not* generic.

  (2) Non-identified model, while clever, is not plausible.
General Hedonic Model: Supply of Workers

- Individual workers match to single worker firms.

- Choose quality of job $z$ and maximize

$$ P(z) - U(z, x, \varepsilon), $$

where $P(z)$ denotes earnings of workers.

- $P(z) = C$ (consumption).
General Hedonic Model: Supply of Workers

- $x$ is vector of observable characteristics of workers with density $f_x(x)$.

- $\varepsilon$ is vector of unobservable worker characteristics with density $f_\varepsilon(\varepsilon)$, independent of $x$.

- Assume $U_{z\varepsilon} < 0$.

- Assume dimension $\varepsilon = \text{dimension of } z$. 
Workers’ Optimization

- FOC:
  \[ P_z (z) - U_z (z, x, \varepsilon) = 0 \]

- SOC:
  \[ P_{zz} - U_{zz} < 0 \]
Choose quality $z$ to maximize output minus cost:

$$\Pi(z) = \Gamma(z, y, \eta) - P(z)$$

- $y$ is vector of observable firm attributes with density $f_y(y)$.

- $\eta$ is a vector of unobservable firm attributes with density $f_\eta(\eta)$, independent of $y$.

- Assume dimension of $\eta = \text{dimension of } Z$.

- Assume $\Gamma_{z\eta} > 0$.  

Firms’ Technologies
Firms’ Optimization

- **FOC:**
  \[
  \Gamma_z(z, y, \eta) - P_z(z) = 0
  \]

- **SOC:**
  \[
  \Gamma_{zz} - P_{zz} < 0
  \]
Goals of Hedonic Analysis

- Analyze first order conditions:

  Workers \[ P_z(z) = U_z(z, x, \varepsilon) \]

  Firms \[ P_z(z) = \Gamma_z(z, y, \eta) \]

- Solve theoretical problem: Characterize equilibrium pricing function.

- Solve econometric problem: Given data on \( (P(z), z, x) \) (or equivalently for firm side) and recover estimates of \( U_z \) and the distribution of \( \varepsilon \) and \( \Gamma_z(z, y, \eta) \) and distribution of \( \eta \).
Workers’ Sorting Conditions

- For each worker \((x, \varepsilon)\), FOC implicitly defines quality supplied (or location chosen): \(z = s(x, \varepsilon)\) (the mapping from worker \((x, \varepsilon)\) to location \(z\)).

- Define the inverse mapping

\[
\varepsilon = \tilde{s}(z, x) .
\]

- Note that \(\tilde{s}(z, x)\) depends on the function \(P\) and

\[
\frac{\partial \tilde{s}}{\partial z} = \frac{P_{zz} - U_{zz}}{U_{z\varepsilon}} > 0 .
\]
Firms’ Sorting Conditions

- For every firm \((y, \eta)\), FOC implicitly defines the quality demanded,
  \[ z = d \left( y, \eta \right), \]
  a mapping from firm \((y, \eta)\) to location \(z\).

- Define the inverse mapping
  \[ \eta = \tilde{d} \left( z, y \right). \]

- \(\tilde{d} \left( z, y \right)\) also depends on \(P\) and
  \[ \frac{\partial \tilde{d}}{\partial z} = \frac{P_{zz} - \Gamma_{zz}}{\Gamma_{z\eta}} > 0. \] (2)
Supply and Demand

- The supply density is
  \[
  \int_{\tilde{X}} f_\varepsilon (\tilde{s}(z, x)) \cdot \frac{\partial \tilde{s}(z, x)}{\partial z} \cdot f_x (x) \, dx.
  \]

- The demand density is
  \[
  \int_{\tilde{Y}} f_\eta (\tilde{d}(z, y)) \cdot \frac{\partial \tilde{d}(z, y)}{\partial z} \cdot f_y (y) \, dy.
  \]
Equilibrium in a Hedonic Market

- Price function equates supply and demand at all $z$:

$$\int_{\tilde{X}} f_\varepsilon (\tilde{s} (z, x)) \frac{\partial \tilde{s} (z, x)}{\partial z} f_x (x) \, dx =$$

$$\int_{\tilde{Y}} f_\eta (\tilde{d} (z, y)) \frac{\partial \tilde{d} (z, y)}{\partial z} f_y (y) \, dy \quad (3)$$

- SOC for all firms and workers are satisfied (local condition).

- $z = s (x, \varepsilon)$ and $z = d (y, \varepsilon)$ are globally optimal (this is the Monge-Ampere differential equation).
Importance of (3)

- Compute equilibria of sample economies.
- Understand how primitives of model influence sorting and pricing.
- Relations between curvature of pricing function and curvatures of preferences and technology.
- Understand when bunching occurs (concentration at particular points of quality).
- Analyze identification of model.
Curvature of Pricing Function: General Nonadditive Case

- Substitute (1) and (2) into (3):

\[
P_{zz} = \frac{\int_{\tilde{Y}} \left( \frac{\Gamma_{zz}}{\Gamma_{z\eta}} \right) f_\eta f_y dy - \int_{\tilde{X}} \left( \frac{U_{zz}}{U_{z\varepsilon}} \right) f_\varepsilon f_x dx}{\left( \int_{\tilde{Y}} \frac{f_\eta f_y}{\Gamma_{z\eta}} dy - \int_{\tilde{X}} \frac{f_\varepsilon f_x}{U_{z\varepsilon}} dx \right)}
\]

- \(P_{zz}\) is a weighted average of curvature of technology and preferences.

- Curvature of technology and preferences only matters at points actually chosen, i.e. \(d(y, \eta)\) and \(s(x, \varepsilon)\).
Special Case 1: Linear FOC

Workers

\[ P_z(z) = U_{zz}z + U_{zx}x - \varepsilon \]

Firms

\[ P_z(z) = \Gamma_{zz}z + \Gamma_{zy}y + \eta \]

- \( \Gamma_{zz} \) and \( U_{zz} \) are constants.

- \( \Gamma_{z\eta} = 1 \) and \( U_{z\varepsilon} = -1 \).

- \( \Gamma_{zy} \) and \( U_{zx} \) are constants.
Special Case 1: Linear FOC

- $P_{zz}$ is a simple weighted average of $\Gamma_{zz}$ and $U_{zz}$.

$$P_{zz} = \frac{\Gamma_{zz} \int_{\tilde{Y}} f_\eta f_y \, dy + U_{zz} \int_{\tilde{X}} f_\varepsilon f_x \, dx}{\int_{\tilde{Y}} f_\eta f_y \, dy + \int_{\tilde{X}} f_\varepsilon f_x \, dx}$$
Special Case 1: Linear FOC

- When all consumer and firm characteristics are distributed normally this simplifies even further to

\[ P_{zz} = \frac{\Gamma_{zz}\sigma_w + U_{zz}\sigma_f}{\sigma_f + \sigma_w}. \]


- In this very special (nongeneric) case the price function is quadratic.
worker side of the market (preferences iso-utility curve)

firm side (iso-profit curve)
Special Case 2: Additive FOC

Workers \[ P_z (z) = m_w (z) + n_w (x) - \varepsilon \]

Firms \[ P_z (z) = m_f (z) + n_f (y) + \eta \]

- \( \Gamma_{zz}, \Gamma_{zy}, U_{zz}, \) and \( U_{zx} \) are not constants.
  
1. \( \Gamma_{z\eta} = 1 \) and \( U_{z\varepsilon} = -1 \).

2. \( \Gamma_{zy} = 0 \) and \( U_{zx} = 0 \).
Special Case 2: Additive FOC

\[ P_{zz} = \frac{\int_{\tilde{Y}} \Gamma_{zz} f_{\eta} f_y dy + \int_{\tilde{X}} U_{zz} f_{\varepsilon} f_x dx}{\int_{\tilde{Y}} f_{\eta} f_y dy + \int_{\tilde{X}} f_{\varepsilon} f_x dx} \]
Bunching

- Classical hedonic model assumes equilibrium sorting of agents is smooth.

- No location has positive mass of people.

- Equilibrium pricing function is $C^2$.

- Are there conditions on primitives that rule out bunching?
SOC of Consumer

- For every consumer who chooses \( z \), it must be the case that \( U_{zz} (z, x, \varepsilon) > P_{zz} (z) \).

\[
\int_{\tilde{Y}} \left( \frac{\Gamma_{zz}}{\Gamma_{z\eta}} \right) f_\eta f_y dy - \int_{\tilde{X}} \left( \frac{U_{zz}}{U_{z\varepsilon}} \right) f_\varepsilon f_x dx < \int_{\tilde{Y}} \left( \frac{f_\eta f_y}{\Gamma_{z\eta}} \right) dy - \int_{\tilde{X}} \left( \frac{f_\varepsilon f_x}{U_{z\varepsilon}} \right) dx
\]

- For every firm who optimally chooses \( z \), it must be the case that

\[
\Gamma_{zz} (z, y, \eta) < P_{zz} (z).
\]
SOC of Firm

\[ \Gamma_{zz} \left( z, y, \tilde{d}(z, y) \right) < \frac{\int_{\tilde{Y}} \left( \frac{\Gamma_{zz}}{\Gamma_{z\eta}} \right) f_\eta f_y \, dy - \int_{\tilde{X}} \left( \frac{U_{zz}}{U_{z\varepsilon}} \right) f_\varepsilon f_x \, dx}{\left( \int_{\tilde{Y}} f_\eta f_y \, dy - \int_{\tilde{X}} f_\varepsilon f_x \, dx \right)} \]

\[ \Gamma_{zz} \left( z, y, \tilde{d}(z, y) \right) < U_{zz} (z, x, \tilde{s}(z, x)) \]
SOC of Firm

- Conditions depend in complicated way on curvatures of preferences and technology and on distribution of preferences and technology.

- In the special case of an additive FOC,

\[ \Gamma_{zz}(z) < U_{zz}(z). \]
No Bunching Example

- Consumer problem:

  \[
  \text{Max}_z \ P(z) - \frac{z^\beta}{\varepsilon}
  \]

  \[
  \varepsilon_l \leq \varepsilon \leq \varepsilon_u
  \]

- Firm problem:

  \[
  \text{Max}_z \ z^\alpha \eta - P(z)
  \]

  \[
  \eta_l \leq \eta \leq \eta_u
  \]
No Bunching Example

- FOC and SOC for the consumer’s problem:

  \[ FOC : P_z - \frac{\beta \, z^{\beta-1}}{\varepsilon} = 0 \]

  \[ SOC : P_{zz} \, \frac{\beta (\beta - 1) \, z^{\beta-2}}{\varepsilon} < 0 \]

- FOC and SOC for the firm’s problem:

  \[ FOC : \alpha \, z^{\alpha-1} \eta - P_z = 0 \]

  \[ SOC : \alpha (\alpha - 1) \, z^{\alpha-2} \eta - P_{zz} < 0 \]
No Bunching Example

- **Inverse supply and demand:**
  \[
  \varepsilon = \tilde{s}(z) = \frac{\beta z^{\beta-1}}{P_z} \quad \text{and} \quad \eta = \tilde{d}(z) = \frac{P_z}{\alpha z^{\alpha-1}}.
  \]

- **Equilibrium condition:**
  \[
  F_\varepsilon \left( \frac{\beta z^{\beta-1}}{P_z(z)} \right) = F_\eta \left( \frac{P_z(z)}{\alpha z^{\alpha-1}} \right),
  \]
  \[
  \frac{\beta z^{\beta-1}}{P_z(z)} = \frac{P_z(z) z^{1-\alpha}}{\alpha},
  \]
  for \( \varepsilon_l \leq \frac{\beta z^{\beta-1}}{P'(z)} \leq \varepsilon_u \).
No Bunching Example

• Equilibrium price function:

\[ P_z(z) = \left( \alpha \beta z^{\alpha + \beta - 2} \right)^{1/2}, \]

where

\[ \varepsilon_l^\frac{2}{\beta - \alpha} \left( \frac{\alpha}{\beta} \right)^\frac{1}{\beta - \alpha} \leq z \leq \varepsilon_u^\frac{2}{\beta - \alpha} \left( \frac{\alpha}{\beta} \right)^\frac{1}{\beta - \alpha}. \]
No Bunching Example

- Supply function:

\[ z = \left( \frac{\alpha}{\beta} \right)^{\frac{1}{\beta-\alpha}} \epsilon^{\frac{2}{\beta-\alpha}}, \]

for \( \epsilon_l \leq \epsilon \leq \epsilon_u \).

- Demand function:

\[ z = \left( \frac{\alpha}{\beta} \right)^{\frac{1}{\beta-\alpha}} \eta^{\frac{2}{\beta-\alpha}}, \]

for \( \eta_l \leq \eta \leq \eta_u \) and \( \alpha < \beta \).
An Example of Equilibrium with Bunching

- $\Gamma(z, \eta) = z^\alpha \eta$, where $\alpha = 0.5$ and $\eta \sim U(0, 1)$.

- $\max_z z^\alpha \eta - P(z)$.

- FOC: $\alpha z^{\alpha - 1} \eta - P_z(z) = 0$.

- SOC: $\alpha(\alpha - 1)z^{\alpha - 2} \eta - P_{zz}(z) < 0$
An Example of Equilibrium with Bunching

- FOC implies
  \[ \eta(z) = \frac{P_z(z) \, z^{1-\alpha}}{\alpha}. \]

- Consumer has a disutility of \( z \) given by
  \[ V(z, \varepsilon) = z^\varepsilon, \]
  where \( \varepsilon \) is a random variable distributed \( U(.25, .75) \).
An Example of Equilibrium with Bunching

- Max$_z$ $P(z) - V(z, \varepsilon)$.

- FOC: $P_z(z) - \varepsilon z^{\varepsilon-1} = 0$.

- SOC: $P_{zz}(z) - \varepsilon (\varepsilon - 1) z^{\varepsilon-2} < 0$.

- $\Gamma_{zz}(z, \eta(z)) < V_{zz}(z, \varepsilon(z))$

- $\Gamma_{zz}(z(\eta), \eta) < V_{zz}(z(\varepsilon), \varepsilon)$.

- $z(\eta) = z(\varepsilon) = z$. 
An Example of Equilibrium with Bunching

- SOC are satisfied if and only if

\[ \alpha (\alpha - 1) z^{\alpha - 2} \eta < \varepsilon (\varepsilon - 1) z^{\varepsilon - 2}. \]

- The last condition becomes:

\[\begin{align*}
(1) & \quad (\alpha - 1) z^{-1} P_z(z) < (\varepsilon - 1) z^{-1} P_z(z) \\
(2) & \quad \alpha < \varepsilon \\
\end{align*}\]

- \[
z = \left(1 - \frac{\alpha}{2 \varepsilon}\right)^{\frac{1}{\varepsilon - \alpha}}
\]
An Example of Equilibrium with Bunching

- \( \eta \) satisfies
  \[
  \eta = \frac{\varepsilon}{\alpha} \left( 1 - \frac{\alpha}{2\varepsilon} \right)
  \]

- Firms with \( \eta < .5 \) and \( \varepsilon < .5 \) will locate at \( z = 0 \).
Consumer Side with Vector $z$

- Preferences quadratic in $z$ and linear in $P(z)$ (ignore $x$):
  \[ U(c, z, \varepsilon, A) = P(z) + \varepsilon'z - \frac{1}{2}z'Az \]

- FOC:
  \[ \varepsilon - Az + P_z = 0 \quad (4) \]

- SOC:
  \[ (P_{zz'} - A) \text{ is negative definite.} \]

- Worker heterogeneity: $\varepsilon \sim N(\mu_\varepsilon, \Sigma_\varepsilon)$.  

Linear-Quadratic Example (Tinbergen, 1956)
Linear-Quadratic Example (Tinbergen, 1956)

**Firm Side**

- Production quadratic in $z$:
  \[
  \Pi(z, \eta, B, P(z)) = \eta'z - \frac{1}{2}z' Bz - P(z) \quad (5)
  \]

- FOC:
  \[
  \eta - Bz - P_z = 0 \quad (6)
  \]

- SOC:
  \[-(B + P_{zz'}) \text{ is negative definite.}\]

- Firm heterogeneity: $\eta \sim N(\mu_\eta, \Sigma_\eta)$. 
Equilibrium

- Equilibrium must satisfy (3).

- Given special structure, one can guess (correctly) that

\[ P(z) = \pi_0 + \pi_1'z + \frac{1}{2}z'\pi_2z. \]

- Then check that the guess is correct.

- Firm FOC:  \[ \eta - Bz - \pi_1 - \pi_2z = 0. \]

- Consumer FOC:  \[ \epsilon - Az + \pi_1 + \pi_2z = 0. \]
Sorting Conditions

\[ z_D = (B + \pi_2)^{-1} (\eta - \pi_1) \]

\[ z_S = (A - \pi_2)^{-1} (\varepsilon + \pi_1) \]

- Equate average demand to average supply:

\[ E^D (z) = (B + \pi_2)^{-1} E (\eta - \pi_1) \]

\[ E^S (z) = (A - \pi_2)^{-1} E (\varepsilon + \pi_1) \]

- One vector equation in unknown coefficients:

\[ (B + \pi_2)^{-1} (\mu_\eta - \pi_1) = (A - \pi_2)^{-1} (\mu_\varepsilon + \pi_1) \]
Sorting Conditions

\[ V^D (z) = (B + \pi_2)^{-1} \sum \eta (B + \pi_2)^{-1}' \]

\[ V^S (z) = (A - \pi_2)^{-1} \sum \varepsilon (A - \pi_2)^{-1}' \]

- Second matrix equation:
  \[ (A - \pi_2)^{-1} \sum \varepsilon (A - \pi_2)^{-1}' = (B + \pi_2)^{-1} \sum \eta (B + \pi_2)^{-1}' \]

- Initial conditions:
  \[ U(z) \geq U_0 \quad \text{and} \quad \Pi(z) \geq 0. \]
Sorting Conditions

- This implies $\pi_0 = 0$.

- Solution depends on

  1. Production and preference parameters $A, B$.

  2. Heterogeneity $\mu_\eta, \mu_\varepsilon, \Sigma_\eta$, and $\Sigma_\varepsilon$. 
Sorting Conditions

- Except in polar cases, price function does not directly reveal any individual structural parameters.

- Note that equilibrium matching implies

\[(B + \pi_2)^{-1}(\eta - \pi_1) = (A - \pi_2)^{-1}(\varepsilon + \pi_1).\]  

- Functional and statistical dependence between \(\eta\) and \(\varepsilon\).
Identifying and Estimating the Tinbergen Model

- Widely used two step estimate procedure (Rosen):

  1. Estimate $P(z)$ from market data.

  2. Use first-order conditions (4) and (6) in conjunction with the marginal prices obtained from step 1 to recover preferences and technology respectively.

- $\hat{\pi}_1$ and $\hat{\pi}_2$ are fitted price coefficients.
Identifying and Estimating the Tinbergen Model

- $x$ and $y$ are observable worker and firm regressors:

\[
\hat{\pi}_1 + \hat{\pi}_2 z = \mu_\eta (y) - Bz + \omega_\eta \\
\hat{\pi}_1 + \hat{\pi}_2 z = -\mu_\varepsilon (x) + Az - \omega_\varepsilon
\]  

(8)  

(9)

- $\omega_\eta$ and $\omega_\varepsilon$ are unobservable:

\[
\omega_\eta = \eta - \mu_\eta (y) \\
\omega_\varepsilon = \varepsilon - \mu_\varepsilon (x)
\]
Claim One: Identification Can Only be Obtained Through Arbitrary Functional Form Assumptions

- If fitted $P(z)$ is quadratic, linear functions of $z$ on left and right sides.

- If fitted $P(z)$ is not quadratic, nonlinearity can help with identification.

- This nonlinearity is arbitrary.
Claim One: Identification Can Only be Obtained Through Arbitrary Functional Form Assumptions

- However, small perturbations of above model lead to non-quadratic $P(z)$.

- Economics of the problem suggests that for many applications this quadratic equilibrium model not very good any way we want to move away from Tinbergen specification.

- Analysis of equilibrium equation shows that in fact, nonquadratic $P(z)$ is generic.
Example 1

- Perturb scalar version of quadratic model.

- Suppose heterogeneity distributed as mixture of normals with weights (0.999, 0.001) or (0.99, 0.01).

- Marginal price function is clearly nonlinear.

- Unattractive features of quadratic model:
  1. Negative and positive $z$.
  2. Negative marginal products.
Example 2

- Restrict marginal product to be positive.

- Restrict marginal utility of work to be negative.

- The non-identified case is uninteresting as well as unlikely.
Claim Two: Endogeneity Problem

- \( z \) is endogenous in (8) and (9).

- Rewrite (7) as

  \[
  \omega_\eta = \omega_\varepsilon + (A - B) z + \mu_\varepsilon (x) - \mu_\eta (y).
  \]

- Conditional on \( z \), there is a functional and statistical dependence between \( x, y, \omega_\eta, \) and \( \omega_\varepsilon \).
Claim Three: Use of Multimarket Data

- Provides variation not available in a single cross section.

- But why do prices vary across markets?

- Must specify which elements vary across markets, and which elements do not.

- Our results show there is no need for multimarket data.
Generic Identification of General Additive Scalar Model

- Establish points made in previous section more formally and more generally:


  2. Recover all structural parameters up to location.

- Results apply to any model in which FOC reduce to additive first-order conditions.

- Generalization of quadratic model.

- Can constrain to have economically meaningful interpretations.
### Model 1, Unrestricted $z$

<table>
<thead>
<tr>
<th>Components</th>
<th>1</th>
<th>2</th>
</tr>
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<tbody>
<tr>
<td>$\mu_{\nu_1}$</td>
<td>-1.0</td>
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<tr>
<td>$\mu_{\theta_1}$</td>
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<tr>
<td>$\sigma^2_{\nu_1}$</td>
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<tr>
<td>$\sigma^2_{\theta_1}$</td>
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Table 2
Model 2
Linear Quadratic Technologies
Non-Negative $z$

<table>
<thead>
<tr>
<th>Firms</th>
<th>$\Pi(z) = \nu_0 + \nu_1 z - \frac{1}{2}bz^2 - p(z)$</th>
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<tr>
<td></td>
<td>$\nu_1, b \geq 0$</td>
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<tr>
<td></td>
<td>$\ln \nu_1 = \nu_{10} + \nu'_{11} x + \varepsilon_1$</td>
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<tr>
<td></td>
<td>$x$ and $\varepsilon_1$ are both distributed as a mixture of normals</td>
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<tr>
<td></td>
<td>(the mixtures could have only one component).</td>
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<table>
<thead>
<tr>
<th>FOC</th>
<th>$\nu_1 - bz - p'(z) = 0$</th>
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<tbody>
<tr>
<td>SOC</td>
<td>$-b - p''(z) &lt; 0$</td>
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</table>

<table>
<thead>
<tr>
<th>Workers</th>
<th>$V(z) = \theta_0 + \theta_1 z - \frac{1}{2}az^2 + p(z)$</th>
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<tr>
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<td>$\theta_1, a \geq 0$</td>
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<td>$\ln \theta_1 = \theta_{10} + \theta'_{11} y + \varepsilon_2$</td>
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<td></td>
<td>$y$ and $\varepsilon_2$ are both distributed as mixtures of normals</td>
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<tr>
<td>Components</td>
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<td>$\lambda_\theta$</td>
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Figure 1: Slope of Price Function - Model 1a
Figure 2: Curvature of Price Function - Model 1a

\[ P''(z) \]

\[ \lambda = 1.00 \]

\[ \lambda = 0.999 \]
Figure 3: Slope of Price Function - Model 1b

$\lambda = 0.99$

$\lambda = 0.90$

$P''(z)$
Figure 4: Curvature of Price Function - Model 1b
Figure 5: Slope of Pricing Function - Model 2

![Graph showing the slope of a pricing function over a range of values for variable z. The graph indicates a positive and increasing trend.]
Figure 6: Curvature of Price Function - Model 2
<table>
<thead>
<tr>
<th>Components</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_\nu$</td>
<td>0.0</td>
<td>1.0</td>
</tr>
<tr>
<td>$\mu_\theta$</td>
<td>1.0</td>
<td>-1.0</td>
</tr>
<tr>
<td>$\sigma^2_\nu$</td>
<td>0.5</td>
<td>1.0</td>
</tr>
<tr>
<td>$\sigma^2_\theta$</td>
<td>1.0</td>
<td>0.1</td>
</tr>
</tbody>
</table>
Figure 1: Slope of Price Function: Model 1
Figure 2: Curvature of Price Function: Model 1

$p''(z)$

Density of $z$

$\lambda = 0.5$

$\lambda = 0.9$

$\lambda = 1$
Figure 3: Slope of Price Function: Model 2

\[ p'(z) \]

Density of \( z \)

- \( \lambda = 0.5 \)
- \( \lambda = 0.9 \)
- \( \lambda = 1 \)
Figure 4: Elasticity of Slope of Price Function with Respect to $z$

- Elasticity
- Density of $z$

- $\lambda = 0.5$
- $\lambda = 0.9$
- $\lambda = 1$
<table>
<thead>
<tr>
<th>Model 1, Unrestricted $z$</th>
<th>Components</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
<tr>
<td>$\mu_{\nu_1}$</td>
<td>-1.0</td>
</tr>
<tr>
<td>$\mu_{\theta_1}$</td>
<td>1.0</td>
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<tr>
<td>$\sigma^2_{\nu_1}$</td>
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<tr>
<td>$\sigma^2_{\theta_1}$</td>
<td>0.5</td>
</tr>
</tbody>
</table>
### Table 2
#### Model 2
Linear Quadratic Technologies
Non-Negative $z$

<table>
<thead>
<tr>
<th>Firms</th>
<th>$\Pi(z) = \nu_0 + \nu_1 z - \frac{1}{2} b z^2 - p(z)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\nu_1, b \geq 0$</td>
</tr>
<tr>
<td></td>
<td>$\ln \nu_1 = \nu_{10} + \nu_{11} x + \varepsilon_1$</td>
</tr>
<tr>
<td></td>
<td>$x$ and $\varepsilon_1$ are both distributed as a mixture of normals (the mixtures could have only one component).</td>
</tr>
<tr>
<td>FOC</td>
<td>$\nu_1 - b z - p'(z) = 0$</td>
</tr>
<tr>
<td>SOC</td>
<td>$-b - p''(z) &lt; 0$</td>
</tr>
<tr>
<td>Workers</td>
<td>$V(z) = \theta_0 + \theta_1 z - \frac{1}{2} a z^2 + p(z)$</td>
</tr>
<tr>
<td></td>
<td>$\theta_1, a \geq 0$</td>
</tr>
<tr>
<td></td>
<td>$\ln \theta_1 = \theta_{10} + \theta'_{11} y + \varepsilon_2$</td>
</tr>
<tr>
<td></td>
<td>$y$ and $\varepsilon_2$ are both distributed as mixtures of normals</td>
</tr>
</tbody>
</table>
## Model 2

<table>
<thead>
<tr>
<th>Components</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_\nu$</td>
<td>1.0</td>
<td>2.0</td>
</tr>
<tr>
<td>$\mu_\theta$</td>
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<td>0.5</td>
</tr>
<tr>
<td>$\sigma^2_\nu$</td>
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<td>0.21</td>
</tr>
<tr>
<td>$\sigma^2_\theta$</td>
<td>0.61</td>
<td>0.61</td>
</tr>
<tr>
<td>$\lambda_\nu$</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>$\lambda_\theta$</td>
<td>0.5</td>
<td>0.5</td>
</tr>
</tbody>
</table>
Figure 1: Slope of Price Function - Model 1a
Figure 2: Curvature of Price Function - Model 1a

The graph shows the curvature of the price function for two models, 

- Model 1 with \( \lambda = 1.00 \)
- Model 1 with \( \lambda = 0.999 \)

The graph indicates the behavior of the price function \( P''(z) \) as \( z \) varies from \(-3\) to \(2\).
Figure 3: Slope of Price Function - Model 1b

- $\lambda = 0.99$
- $\lambda = 0.90$
Figure 4: Curvature of Price Function - Model 1b
Figure 5: Slope of Pricing Function - Model 2
Figure 6: Curvature of Price Function - Model 2
<table>
<thead>
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<th>Components</th>
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<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_\nu$</td>
<td>0.0</td>
<td>1.0</td>
</tr>
<tr>
<td>$\mu_\theta$</td>
<td>1.0</td>
<td>-1.0</td>
</tr>
<tr>
<td>$\sigma^2_\nu$</td>
<td>0.5</td>
<td>1.0</td>
</tr>
<tr>
<td>$\sigma^2_\theta$</td>
<td>1.0</td>
<td>0.1</td>
</tr>
</tbody>
</table>
Figure 1: Slope of Price Function: Model 1

The top graph shows the slope of the price function, $p'(z)$, for different values of $\lambda$. The bottom graph illustrates the density function of $z$ for the same values of $\lambda$. The graphs are labeled with $\lambda = 0.5$, $\lambda = 0.9$, and $\lambda = 1$. The y-axis in the top graph represents the slope of the price function, while the y-axis in the bottom graph represents the density of $z$. The x-axis in both graphs represents the variable $z$. The graphs display the behavior of the price function and its density under varying conditions of $\lambda$. 
Figure 2: Curvature of Price Function: Model 1

\[
\lambda = 0.5
\]

\[
\lambda = 0.9
\]

\[
\lambda = 1
\]
Figure 3: Slope of Price Function: Model 2

\[ p'(z) \]

\[ \lambda = 0.5 \]
\[ \lambda = 0.9 \]
\[ \lambda = 1 \]
Figure 4: Elasticity of Slope of Price Function with Respect to $z$

- $\lambda = 0.5$
- $\lambda = 0.9$
- $\lambda = 1$