Notes on Generalized Roy Model

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\[ Y_1 = \mu_1(X) + U_1 \]
\[ Y_0 = \mu_0(X) + U_0 \]
\[ C = \mu_C(Z) + U_C \]

Net Benefit: \[ I = Y_1 - Y_0 - C \]

\[ I = \mu_1(X) - \mu_0(X) - \mu_C(Z) + \underbrace{U_1 - U_0 - U_C}_{\mu_D(Z)} - V \]

\((U_0, U_1, U_C) \perp \perp (X, Z)\)

\[ E(U_0, U_1, U_C) = (0, 0, 0) \]

\( V \perp \perp (X, Z) \)
Assume Normally Distributed Errors.

Assume $Z$ contains $X$ but may contain other variables (exclusions)

$$Y = DY_1 + (1 - D)Y_0$$

observed $Y$

$$D = 1(I \geq 0) = 1(\mu_D(Z) \geq V)$$

Assume $V \sim N(0, \sigma^2_V)$
Propensity Score:

\[
\Pr(D = 1 \mid Z = z) = \Phi \left( \frac{\mu_D(z)}{\sigma_V} \right)
\]

\[
E(Y \mid D = 1, X = x, Z = z) = \mu_1(X) + E(U_1 \mid \mu_D(z) \geq V) \tag{K_1(P(z))}
\]

because \((X, Z) \perp \perp (U_1, V)\).

Under normality we obtain

\[
E \left( U_1 \left| \frac{\mu_D(z)}{\sigma_V} \geq \frac{V}{\sigma_V} \right. \right) = \frac{\text{Cov}(U_1, \frac{V}{\sigma_V})}{\text{Var}(\frac{V}{\sigma_V})} \lambda \left( \frac{\mu_D(z)}{\sigma_V} \right)
\]
Why?

\[ U_1 = \text{Cov} \left( U_1, \frac{V}{\sigma_V} \right) \frac{V}{\sigma_V} + \varepsilon_1 \]

\[ \varepsilon_1 \perp V \]

\[ E \left( \frac{V}{\sigma_V} \mid \frac{\mu_D(z)}{\sigma_V} \geq \frac{V}{\sigma_V} \right) = \frac{\mu_D(z)}{\sigma_V} \int_{-\infty}^{\mu_D(z)} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \, dt \]

\[ = \frac{-1}{\sqrt{2\pi}} e^{\left(-\frac{1}{2}\right)\left(\frac{\mu_D(z)}{\sigma_V}\right)^2} \]

\[ = \frac{-\phi \left( \frac{\mu_D(z)}{\sigma_V} \right)}{\Phi \left( \frac{\mu_D(z)}{\sigma_V} \right)} = \tilde{\lambda} \left( \frac{\mu_D(z)}{\sigma_V} \right) = \phi \left( \frac{\mu_D(z)}{\sigma_V} \right) \]
Notice

\[
\lim_{\mu_D(z) \to \infty} \tilde{\lambda} \left( \frac{\mu_D(z)}{\sigma_V} \right) = 0
\]

\[
\lim_{\mu_D(z) \to -\infty} \tilde{\lambda} \left( \frac{\mu_D(z)}{\sigma_V} \right) = -\infty
\]

Propensity score:

\[
P(z) = \Pr(D = 1 \mid Z = z) = \Phi \left( \frac{\mu_D(z)}{\sigma_V} \right)
\]

\[
\therefore \left( \frac{\mu_D(z)}{\sigma_V} \right) = \Phi^{-1} (\Pr(D = 1 \mid Z = z))
\]
Thus we can replace $\frac{\mu_D(z)}{\sigma_V}$ with a known function of $P(z)$.

As $P(Z) \to 1$, selection bias term goes to zero.
Notice that because $(X, Z) \perp (U, V)$, $Z$ enters the model (conditional on $X$) only through $P(Z)$.

This is called index sufficiency.

We can apply our material on the Roy model to the Generalized Roy model.
Put all of these results together to obtain

\[
E(Y \mid D = 1, X = x, Z = z) = \mu_1(x) + \left( \frac{\text{Cov}(U_1, \frac{V}{\sigma_V})}{\text{Var}(\frac{V}{\sigma_V})} \right) \tilde{\lambda} \left( \frac{\mu_D(z)}{\sigma_V} \right)
\]

\[
= E(Y_1 \mid D = 1, X = x, Z = z) = \mu_1(x) + \left( \frac{\text{Cov}(U_1, \frac{V}{\sigma_V})}{\text{Var}(\frac{V}{\sigma_V})} \right) \tilde{\lambda} \left( \frac{\mu_D(z)}{\sigma_V} \right)
\]

\[
\tilde{\lambda}(z) = E \left( \frac{V}{\sigma_V} \mid \frac{V}{\sigma_V} < \frac{\mu_D(z)}{\sigma_V} \right) < 0
\]

\[
\lambda(z) = E \left( \frac{V}{\sigma_V} \mid \frac{V}{\sigma_V} \geq \frac{\mu_D(z)}{\sigma_V} \right) > 0
\]

\[
E(Y \mid D = 0, X = x, Z = z) = \mu_0(x) + \left( \frac{\text{Cov}(U_0, \frac{V}{\sigma_V})}{\text{Var}(\frac{V}{\sigma_V})} \right) \lambda \left( \frac{\mu_D(z)}{\sigma_V} \right)
\]

\[
\text{Var} \left( \frac{V}{\sigma_V} \right) = 1
\]
\[ \frac{V}{\sigma_V} = -\frac{(U_1 - U_0 - U_C)}{\sigma_V} \]

\[ \text{Cov} \left( U_1, \frac{V}{\sigma_V} \right) = -\text{Cov} \left( U_1, \frac{V}{\sigma_V} \right) + \text{Cov} \left( U_0, \frac{V}{\sigma_V} \right) + \text{Cov} \left( U_C, \frac{V}{\sigma_V} \right) \]

In Roy model case \((U_C = 0)\),

\[ \text{Cov} \left( U_1, \frac{V}{\sigma_V} \right) = -\text{Cov} \left( U_1, \frac{U_1 - U_0}{\sigma_V} \right) \]

\[ = \frac{\text{Cov} (U_1 - U_0, U_1)}{\sqrt{\text{Var}(U_1 - U_0)}} \]
We can identify $\mu_1(x), \mu_0(x)$

From Discrete Choice model we can identify

$$\frac{\mu_D(z)}{\sigma_V} = \frac{\mu_1(x) - \mu_0(x) - \mu_C(z)}{\sigma_V}$$

If we have a regressor in $X$ that does not affect $\mu_C(z)$ (say regressor $x_j$, so $\frac{\partial \mu_C(z)}{\partial x_j} = 0$), we can identify $\sigma_V$ and $\mu_C(z)$.

\[ \therefore \text{ We can identify the net benefit function and the cost function up to scale.} \]

\[ \therefore \text{ We can compute } ex \ ante \text{ subjective net gains.} \]
Method generalizes:
Don’t need normality

\[
E (Y \mid D = 1, X = x, Z = z) = \mu_1(x) + \underbrace{K_1(P(z))}_{\text{control function}}
\]

\[
E (Y \mid D = 0, X = x, Z = z) = \mu_0(x) + \underbrace{K_0(P(z))}_{\text{control function}}
\]

\[
\lim_{P(z) \to 1} E (Y \mid D = 1, X = x, Z = z) = \mu_1(x)
\]

\[
\lim_{P(z) \to 0} E (Y \mid D = 0, X = x, Z = z) = \mu_0(x)
\]

“Identification at infinity”
If this condition satisfied, we can identify ATE:

\[ E(Y_1 - Y_0 \mid X = x) = \mu_1(x) - \mu_0(x) \]

ATE is defined in a limit set. This is true for any model with selection on unobservables (IV; selection models)
What about treatment on the treated?

\[
E(Y_1 - Y_0 \mid D = 1, X = x, Z = z)
\]
(a) From the data, we observe

\[ E(Y_1 \mid D = 1, X = x, Z = z) \]

(b) Can also create it from the model

(c) \( E(Y_0 \mid D = 1, X = x, Z = z) \) is a counterfactual

We know

\[ E(Y_0 \mid D = 0, X = x, Z = z) = \mu_0(x) + \text{Cov} \left( U_0, \frac{V}{\sigma_V} \right) \lambda \left( \frac{\mu_D(Z)}{\sigma_V} \right) \]

(this is data)
(d) We seek

\[ E(Y_0 \mid D = 1, X = x, Z = z) = \mu_0(x) + \text{Cov} \left( U_0, \frac{V}{\sigma_V} \right) \tilde{\lambda} \left( \frac{\mu_D(z)}{\sigma_V} \right) \]

But under normality, we know \( \text{Cov} \left( U_0, \frac{V}{\sigma_V} \right) \)

We know \( \frac{\mu_D(Z)}{\sigma_V} \)

\( \tilde{\lambda}(\cdot) \) is a known function.

Can form \( \tilde{\lambda} \left( \frac{\mu_D(z)}{\sigma_V} \right) \) and can construct counterfactual.
More generally, without normality (but with $(X, Z) \perp \perp (U, V)$)

$$E(Y_1 \mid D = 1, X, Z) = E(Y \mid D = 1, X = x, Z = z) = \mu_1(x) + K_1(P(z))$$

$$E(Y_0 \mid D = 0, X, Z) = E(Y \mid D = 0, X = x, Z = z) = \mu_0(x) + \tilde{K}_0(P(z))$$

where $K_1(P(z)) = E(U_1 \mid D = 1, X = x, Z = z) = E \left( U_1 \mid \frac{\mu_D(z)}{\sigma_V} > \frac{V}{\sigma_V} \right)$

$$\tilde{K}_1(P(z)) = E \left( U_1 \mid \frac{\mu_D(z)}{\sigma_V} > \frac{V}{\sigma_V} \right)$$

$$\tilde{K}_0(P(z)) = E \left( U_0 \mid \frac{\mu_D(z)}{\sigma_V} > \frac{V}{\sigma_V} \right)$$
Use the transformation

\[
\frac{F_V}{\sigma_V} \left( \frac{\mu_D(z)}{\sigma_V} \right) = P(z)
\]

\[
\frac{F_V}{\sigma_V} \left( \frac{V}{\sigma_V} \right) = U_D \quad \text{(a uniform random variable)}
\]

\[
D = 1 \left( \frac{\mu_D(z)}{\sigma_V} \geq \frac{V}{\sigma_V} \right) = 1 \left( P(z) \geq U_D \right)
\]

\[
K_1(P(z)) = E(U_1 \mid P(z) > U_D)
\]

\[
K_1(P(z))P(z) + \tilde{K}_1(P(z))(1 - P(z)) = 0
\]

∴ we can construct \( \tilde{K}_1(P(z)) \)

∴ we can form the counterfactual.
Symmetrically

\[ \tilde{K}_0(P(z)) = E(U_0 \mid P(z) \leq U_D) \]

\[ K_0(P(z)) = E(U_0 \mid P(z) > U_D) \]

\[ (1 - P(z))\tilde{K}_0(P(z)) + P(z)K_0(P(z)) = 0 \]
If we have “identification at infinity,” we can construct

$$E(Y_1 - Y_0 \mid X = x) = \mu_1(x) - \mu_0(x)$$

We can construct TT

$$E(Y_1 - Y_0 \mid D = 1, X = x, Z = z) =$$

$$= [\mu_1(x) + K_1(P(z))] - [\mu_0(x) + K_0(P(z))]$$

We form $$\mu_1(x) + K_1(P(z))$$ from data

We get $$\mu_0(x)$$ from limit set $$P(z) \to 0$$ identifies $$\mu_0(x)$$

We can form $$K_0(P(z)) = -\tilde{K}_0(P(z)) \frac{P(z)}{1-P(z)}$$

. Can construct the desired counterfactual mean.
Notice how we can get EOTM

$$E(Y_1 - Y_0 \mid I = 0, X = x, Z = z)$$

Under normality we have (as a result of independence and normality)

$$E(Y_1 - Y_0 \mid I = 0, X = x, Z = z)$$

$$= \mu_1(x) - \mu_0(x) + E\left(U_1 - U_0 \mid \frac{\mu_D(z)}{\sigma_V} = \frac{V}{\sigma_V}, X = x, Z = z\right)$$

$$= \mu_1(x) - \mu_0(x) + \text{Cov}\left(U_1 - U_0, \frac{V}{\sigma_V}\right) \frac{\mu_D(z)}{\sigma_V}$$

In the Roy model case where $U_C = 0$ but $\mu_C(z) \neq 0$

$$= \mu_1(x) - \mu_0(x) - \sigma_V \left(\frac{\mu_D(z)}{\sigma_V}\right)$$

$$= \mu_1(x) - \mu_0(x) - \mu_D(z)$$

$$= \mu_C(z)$$

(marginal gain = marginal cost)
MTE is

\[ E(Y_1 - Y_0 \mid V = v, X = x, Z = z) = \]

\[ = \mu_1(x) - \mu_0(x) + \text{Cov} \left( U_1 - U_0, \frac{V}{\sigma_V} \right) v \]

EOTM picks \( v = \frac{\mu_D(z)}{\sigma_V} \)

Notice we can use the result that

\[ \frac{\mu_D(z)}{\sigma_V} = F^{-1} \left( \frac{V}{\sigma_V} \right) (P(Z)) \]

\[ V = F^{-1} \left( \frac{V}{\sigma_V} \right)(U_D) \]
EOTM:

\[ E(Y_1 - Y_0 \mid I = 0, X = x, Z = z) = \]

\[ = \mu_1(x) - \mu_0(x) + \text{Cov} \left( U_1 - U_0, \frac{V}{\sigma_V} \right) F^{-1} \left( \frac{V}{\sigma_V} \right) (P(z)) \]

MTE:

\[ E(Y_1 - Y_0 \mid V = v, X = x, Z = z) = \]

\[ = \mu_1(x) - \mu_0(x) + \text{Cov} \left( U_1 - U_0, \frac{V}{\sigma_V} \right) F^{-1} \left( \frac{V}{\sigma_V} \right) (U_D) \]
A useful fact:

\[
P(z) = \Pr(D = 1 \mid Z = z) \\
= \Pr(\mu_D(z) \geq V) \\
= \Pr \left( \frac{\mu_D(z)}{\sigma_V} \geq \frac{V}{\sigma_V} \right)
\]

\[
P(z) = F_{\frac{V}{\sigma_V}} \left( \frac{\mu_D(z)}{\sigma_V} \right)
\]

\[
U_D = F_{\frac{V}{\sigma_V}} \left( \frac{V}{\sigma_V} \right); \quad \text{Uniform}(0, 1)
\]
\[ P(z) = \Pr \left( \frac{F_{\frac{V}{\sigma_V}} \left( \frac{\mu_D(z)}{\sigma_V} \right)}{\sigma_V} \right) \geq \frac{F_{\frac{V}{\sigma_V}} \left( \frac{V}{\sigma_V} \right)}{\sigma_V} \]

\[ = \Pr \left( P(z) \geq U_D \right) \]

\( P(z) \) is the \( p(z)^{th} \) quantile of \( U_D \).
Recall

\[ Y = DY_1 + (1 - D)Y_0 \]
\[ = Y_0 + D(Y_1 - Y_0) \]

Keep \( X \) implicit (condition on \( X = x \))

\[ E(Y \mid Z = z) = E(Y_0) + E(Y_1 - Y_0 \mid D = 1, Z = z)P(z) \]

from law of iterated expectations

\[ = E(Y_0) + E(Y_1 - Y_0 \mid P(z) \geq U_D)P(z) \]

\[ \therefore \text{It depends on } Z \text{ only through } P(Z). \]

\[ E(Y \mid Z = z') = E(Y_0) + E(Y_1 - Y_0 \mid P(z') \geq U_D)P(z') \]
What is $E(Y_1 - Y_0 \mid P(z) \geq U_D)$?

Let the joint density of $(Y_1 - Y_0, U_D)$ be

$$f_{Y_1 - Y_0, U_D}(y_1 - y_0, u_D).$$

It does not depend on $Z$. It will, in general, depend on $X$.

$$E(Y_1 - Y_0 \mid P(z) \geq U_D) = \int_{-\infty}^{\infty} \int_{0}^{\infty} (y_1 - y_0) f_{Y_1 - Y_0, U_D}(y_1 - y_0, u_D) \, du_D \, d(y_1 - y_0) \Pr(P(z) \geq U_D)$$

Now recall that

$$U_D = F\left(\frac{V}{\sigma_V}\right) \left(\frac{V}{\sigma_V}\right).$$

$U_D$ is a quantile of the $V/\sigma_V$ distribution.
By construction, $U_D$ is Uniform(0, 1) (this is the definition of a quantile).

∴ $f_{U_D}(u_D) = 1$.

Also, $\Pr(P(z) \geq U_D) = P(z)$.

Notice, by law of conditional probability,

$$f_{Y_1-Y_0, U_D}(y_1-y_0, u_D) = f_{Y_1-Y_0, U_D}(y_1-y_0 \mid U_D = u_D) f_{U_D}(u_D) = 1.$$
\[
E(Y_1 - Y_0 \mid P(z) \geq U_D)
\]
\[
P(z) \int \int (y_1 - y_0) f_{Y_1 - Y_0, U_D}(y_1 - y_0, u_D) d(y_1 - y_0) du_D
\]
\[
= \frac{P(z)}{P(z)}
\]

\[
E(Y_1 - Y_0 \mid P(z) \geq U_D)
\]
\[
P(z) \int \int (y_1 - y_0) f_{Y_1 - Y_0, U_D}(y_1 - y_0 \mid U_D = u_D) d(y_1 - y_0) du_D
\]
\[
= \frac{P(z)}{P(z)}
\]

\[
P(z) \int 0 E(Y_1 - Y_0 \mid U_D = u_D) du_D
\]
\[
= \frac{P(z)}{P(z)}
\]
\[ \therefore E(Y | Z = z) = E(Y_0) + \int_0^{P(z)} E(Y_1 - Y_0 | U_D = u_D) du_D \]

\[ \frac{\partial E(Y | Z = z)}{\partial P(z)} = E(Y_1 - Y_0 | U_D = P(z)) \]

EOTM or marginal gains

\[ E(Y | Z = z') = E(Y_0) + \int_0^{P(z')} E(Y_1 - Y_0 | U_D = u_D) du_D \]
Suppose $P(z) > P(z')$

\[ \therefore E(Y \mid Z = z) - E(Y \mid Z = z') = \]

\[ = \int_{P(z')}^{P(z)} E(Y_1 - Y_0 \mid U_D = u_D) du_D \]

\[ = E(Y_1 - Y_0 \mid P(z) \geq U_D \geq P(z')) \Pr(P(z) \geq U_D \geq P(z')) \]
Notice

\[ \Pr(P(z) \geq U_D \geq P(z')) = \int_{P(z')}^{P(z)} du_D \]

\[ = P(z) - P(z') \]

\[ E(Y \mid Z = z) - E(Y \mid Z = z') \]

\[ = E(Y_1 - Y_0 \mid P(z) \geq U_D \geq P(z'))(P(z) - P(z')) \]

LATE
\[
\frac{E(Y \mid Z = z) - E(Y \mid Z = z')}{P(z) - P(z')} = \text{LATE}(z, z')
\]

\[
\frac{P(z)}{P(z')} \int \frac{\text{MTE}(u_D)}{P(z')} du_D = \frac{P(z)}{P(z) - P(z')}
\]
Policy Relevant Treatment Effect

(Keep \( X \) implicit)

\[
E(Y_p) = \int_0^1 E(Y_p \mid P_p(Z_p) = t) \, dF_{P_p}(t)
= \int_0^1 \left[ \int_0^1 \left( \begin{array}{c}
1_{[0,t]}(u_D)E(Y_{1,p} \mid U_D = u_D) \\
+ 1_{(t,1]}(u_D)E(Y_{0,p} \mid U_D = u_D)
\end{array} \right) \right] \, du \, dF_{P_p}(t)
= \int_0^1 \left[ \int_0^{u_D} \left( \begin{array}{c}
1_{[0,u_D]}(t)E(Y_{1,p} \mid U_D = u_D) \\
+ 1_{[0,u_D]}(t)E(Y_{0,p} \mid U_D = u_D)
\end{array} \right) \, dt \right] \, du_D
= \int_0^1 \left[ (1 - F_{P_p}(u_D))E(Y_{1,p} \mid U_D = u_D) \\
+ F_{P_p|X}(u_D)E(Y_{0,p} \mid U_D = u_D)
\right] \, du_D.
\]
This derivation involves changing the order of integration.

Note that from finiteness of the mean, 

\[
E \left| 1_{[0,t]}(u_D)E(Y_{1,p} \mid U_D = u_D) + 1_{(t,1]}(u_D)E(Y_{0,p} \mid U_D = u_D) \right| \\
\leq E(\|Y_1\| + \|Y_0\|) < \infty,
\]

\[\therefore\] the change in the order of integration is valid by Fubini’s theorem.
Comparing policy $p$ to policy $p'$,

\[
E(Y_p) - E(Y_{p'}) = \int_0^1 \mathbb{E}(Y_1 - Y_0 \mid U_D = u_D)(F_{P_{p'}}(u_D) - F_{P_p}(u_D)) du_D,
\]

which gives the required weights.

Policies shift the distribution of $P(Z)$.

They keep the distribution of $Y_1$ and $Y_0$ unchanged.