Hedonic Models and Sorting (Nonlinear Pricing)

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Econ 350
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Quality and variety are central features of a modern economy.

(1) As wealth rises, technology improves. It is quality of goods per capita that increases, not mainly their quantity. The same is true of labor force. As scale increases, so does the possibility for comparative advantage and investment in specialized skills which are a source of improvement for modern economies.
Why is the Study of Hedonic Markets Important?

(2) Traditional approach in economics in the mid 1950’s assumes uniform quality of a finite number of goods.

(3) Tinbergen and Theil in mid 1950’s in Rotterdam were among the first to recognize the importance of heterogeneity, variety and quality in economics and how to model it.
Why is the Study of Hedonic Markets Important?

- Economics of quality has a substantial impact on our understanding of:
  
  (1) Consumer Price Indices

  (2) Wage Inequality

  (3) How Regulation and Deregulation Affect Consumer Welfare
Why is the Study of Hedonic Markets Important?

Questions:

1. How to formulate the problem in more general cases?

2. How to make it empirically operational, testable on data and useful for policy analysis?
Acknowledgments

• This presentation draws on:
  1. Tinbergen (1956)
Introduction to Hedonic Markets

- Study competitive hedonic markets, markets for heterogeneous goods in which characteristics of goods are priced out.

- Rosen (1974) proposed an identification strategy to recover preferences and technology.

- Widely believed that parameters of hedonic models are identified only through arbitrary functional form and exclusion assumptions, especially when they are estimated from data on a single market. (Brown and Rosen Critique)
Introduction to Hedonic Markets

Some Conclusions to be Established

- Hedonic model is generically nonlinear.

- This nonlinearity is a source of identification, even in single markets.

- Linearity is an arbitrary and misleading functional form in the context of identifying and estimating hedonic models.
Introduction to Hedonic Markets

- The economic model for which widely used linearization methods are exact is implausible.

- Commonly used linearization strategies produce identification problems.
Introduction to Hedonic Markets

- In a wide class of additive, parametric models, parameters are generically identified with data from a single market.

- In additive nonparametric models, parameters are identified up to affine transformations.

- In general nonlinear models, identified up to monotone transformations.
Introduction to Hedonic Markets

- Identification analysis also applies to other nonlinear pricing models.

  (1) Effects of taxes on behavior when taxes are set optimally (Mirrlees (1971))

  (2) Monopoly pricing (Mussa and Rosen (1978))

  (3) Taxes and labor supply (Heckman (1974); Hausman (1980))

  (4) Social interactions and sorting (Nesheim (2001))

- Analysis can be extended to non-additive models (Heckman, Matzkin, and Nesheim (2002)).
Outline of Lecture

- Presents an hedonic model and review an important quadratic case due to Tinbergen (1956), and used by Epple (1987).

  (1) Equilibrium price depends on production technology parameters, consumer preference parameters, and the distribution of heterogeneity in the population.

  (2) Tinbergen model has closed form solution resulting in quadratic pricing function.
Outline of Lecture: Section 8

- Brown and H. Rosen’s (1982) critique:
  1. Identification depends on arbitrary functional form restrictions.
  2. Attributes $z$ are endogenous; no instruments are available.
  3. Multimarket data can help but caution is required.
Outline of Lecture: Section 8

- Brown-Rosen critique is not valid when quadratic model is perturbed.

  (1) Non-identification is *not* generic.

  (2) Non-identified model, while clever, is not plausible.
General Hedonic Model: Supply of Workers

**Basic Model:**

- Individual workers match to single worker firms.
- Choose quality of job $z$ and maximize
  \[ P(z) - U(z, x, \theta), \]
  where $P(z)$ denotes earnings of workers.
- $z$ and $\theta$ are both scalar.
- $P(z) = C$ (consumption).
General Hedonic Model: Supply of Workers

- $x$ is vector of observable characteristics of workers with density $f_x(x)$.

- $\theta$ is vector of unobservable worker characteristics with density $f_\theta(\theta)$, independent of $x$.

- Assume $U_{z\theta} \neq 0$. 
Workers’ Optimization

- FOC:

\[ P_z(z) - U_z(z, x, \theta) = 0 \]

- SOC:

\[ P_{zz} - U_{zz} < 0 \]
Firms’ Technologies

Firms:

- Choose quality $z$ to maximize output minus cost:
  \[ \Pi(z) = \Gamma(z, y, v) - P(z) \]

- $y$ is vector of observable firm attributes with density $f_y(y)$.

- $v$ is a vector of unobservable firm attributes with density $f_v(v)$, independent of $y$.

- Assume $v$ and $z$ scalar.

- Assume $\Gamma_{zv} \neq 0$. 
Firms’ Optimization

- **FOC:**
  \[ \Gamma_z (z, y, v) - P_z (z) = 0 \]

- **SOC:**
  \[ \Gamma_{zz} - P_{zz} < 0 \]
Goals of Hedonic Analysis

- Analyze first order conditions:
  
  \[ P_z(z) = U_z(z, x, \theta) \]
  
  \[ P_z(z) = \Gamma_z(z, y, \nu) \]

- Solve theoretical problem: Characterize equilibrium pricing function.

- Solve econometric problem: Given data on \((P(z), z, x)\) (or equivalent data for firm side) and recover estimates of \(U_z\) and the distribution of \(\theta\) and \(\Gamma_z(z, y, \nu)\) and distribution of \(\nu\).
Workers’ Sorting Conditions

- For each worker \((x, \theta)\), FOC implicitly defines quality supplied (or location chosen): \(z = s(x, \theta)\) (the mapping from worker \((x, \theta)\) to location \(z\)).

- Define the inverse mapping

\[
\theta = \tilde{s}(z, x).
\]

- Note that \(\tilde{s}(z, x)\) depends on the function \(P\) and for a scalar \(z\) and \(\theta\),

\[
\frac{\partial \tilde{s}}{\partial z} = \frac{P_{zz} - U_{zz}}{U_{z\theta}}. \tag{1}
\]
Firms’ Sorting Conditions

- For every firm \((y, v)\), FOC implicitly defines the quality demanded,
  \[ z = d(y, v), \]
  a mapping from firm \((y, v)\) to location \(z\).

- Define the inverse mapping
  \[ v = \tilde{d}(z, y). \]

- \(\tilde{d}(z, y)\) also depends on \(P\) and for a scalar \(z\) and \(v\),
  \[ \frac{\partial \tilde{d}}{\partial z} = \frac{P_{zz} - \Gamma_{zz}}{\Gamma_{zv}}. \]
Supply and Demand

- The supply density is

\[
\int_{\tilde{x}} f_{\theta}(\tilde{s}(z, x)) \cdot \left| \frac{\partial \tilde{s}(z, x)}{\partial z} \right| \cdot f_{x}(x) \, dx.
\]

- For a fixed \( x = \bar{x} \),
  the supply density is

\[
f_{\theta}(\tilde{s}(z, \bar{x})) \left| \frac{\partial \tilde{s}(z, x)}{\partial z} \right|.
\]
Supply and Demand

- The demand density is

\[
\int_{\tilde{Y}} f_v \left( \tilde{d} (z, y) \right) \cdot \left| \frac{\partial \tilde{d} (z, y)}{\partial z} \right| \cdot f_y (y) \, dy .
\]

- For a fixed \( y = \bar{y} \),
  the demand density is

\[
f_v \left( \tilde{d} (z, \bar{y}) \right) \left| \frac{\partial \tilde{d}(z,\bar{y})}{\partial z} \right|.\]
Equilibrium in a Hedonic Market

- Price function equates supply and demand at all $z$:

$$\int_{\tilde{X}} f_\theta (\tilde{s}(z, x)) \left| \frac{\partial\tilde{s}(z, x)}{\partial z} \right| f_x(x) \, dx = \int_{\tilde{Y}} f_v (\tilde{d}(z, y)) \left| \frac{\partial\tilde{d}(z, y)}{\partial z} \right| f_y(y) \, dy \quad (3)$$

- For a fixed $x = \bar{x}, y = \bar{y}$,

$$f_\theta (\tilde{s}(z, \bar{x})) \left| \frac{\partial\tilde{s}(z, x)}{\partial z} \right| = f_v (\tilde{d}(z, \bar{y})) \left| \frac{\partial\tilde{d}(z, y)}{\partial z} \right|.$$

- SOC for all firms and workers are satisfied (local condition).

- This is the Monge-Ampere differential equation.
Importance of (3)

- Compute equilibria of sample economies.

- Understand how primitives of model influence sorting and pricing.

- Relations between curvature of pricing function and curvatures of preferences and technology.

- Understand when bunching occurs (concentration at particular points of quality).

- Analyze identification of model.
Curvature of Pricing Function: General Nonadditive Case

- Substitute (1) and (2) into (3):

\[
P_{zz} = \frac{\int \left( \frac{\Gamma_{zz}}{\Gamma_{zv}} \right) f_v f_y dy - \int \left( \frac{U_{zz}}{U_{z\theta}} \right) f_{\theta} f_x dx}{\left( \int \frac{f_v f_y}{\Gamma_{zv}} dy - \int \frac{f_{\theta} f_x}{U_{z\theta}} dx \right)}
\]

- \( P_{zz} \) is a weighted average of curvature of technology and preferences.

- Curvature of technology and preferences only matters at points actually chosen, i.e. \( d(y, v) \) and \( s(x, \theta) \).
Special Case 1: Linear FOC

- Assume Linear-quadratic preferences and technology

Workers \( P_z(z) = U_{zz}z + U_{zx}x + \theta \)

Firms \( P_z(z) = \Gamma_{zz}z + \Gamma_{zy}y + v \)

- \( \Gamma_{zz} \) and \( U_{zz} \) are constants.

- \( \Gamma_{zy} = 1 \) and \( U_{z\theta} = 1 \).

- \( \Gamma_{zy} \) and \( U_{zx} \) are constants.
Special Case 1: Linear FOC

- $P_{zz}$ is a simple weighted average of $\Gamma_{zz}$ and $U_{zz}$.

$$P_{zz} = \frac{\Gamma_{zz} \int f_v f_y dy - U_{zz} \int f_0 f_x dx}{\int f_v f_y dy - \int f_0 f_x dx}$$
worker side of the market (preferences iso-utility curve)

firm side (iso-profit curve)
Special Case 2: Additive FOC

Workers  \[ P_z(z) = m_w(z) + n_w(x) + \theta \]

Firms  \[ P_z(z) = m_f(z) + n_f(y) + \nu \]

- \( \Gamma_{zz}, \Gamma_{zy}, U_{zz}, \) and \( U_{zx} \) are \textit{not} constants.

1. \( \Gamma_{zv} = 1 \) and \( U_{z\theta} = 1 \).

2. \( \Gamma_{zy} = 0 \) and \( U_{zx} = 0 \).
Special Case 2: Additive FOC

\[ P_{zz} = \frac{\int_{\tilde{Y}} \Gamma_{zz} f_v f_y dy - \int_{\tilde{X}} U_{zz} f_\theta f_x dx}{\int_{\tilde{Y}} f_v f_y dy - \int_{\tilde{X}} f_\theta f_x dx} \]
Consider a Vector Case where $P(z)$ Plays a More Explicit Role In The Notation

- $P(z)$, earnings of workers with attribute $z$.
- $R$, unearned income.
- $U(c, z, \theta, A)$, preferences of workers
  - $c$, consumption
  - $z$, vector of characteristics
  - $\theta$, heterogeneity parameters in preferences
  - $A$, common preference parameter
- $P(z)$ is hedonic function in $c = P(z) + R$. 
FOC: $U_c(c, z, \theta, A)P_z(z) + U_z(c, z, \theta, A) = 0$.

SOC: $(U_{zz'} + U_cP_{zz'} + P_zU_{cc'}(P_z)')$ is negative definite.

Output $F(z; v, B)$

Production costs $P(z)$.

$B$ is a common technology parameter:

$$\Pi(z, v, B, P(z)) = F(z; v, B) - P(z)$$
FOC:
\[ F_z(z, v, B) - P_z(z) = 0. \] (4)

SOC:
\[ (F_{zz'} - P_{zz'}) \] is negative definite. (5)

\( f_\theta \) and \( f_v \) are the densities of \( \theta \) and \( v \).

Assume that \( \dim(\theta) \geq \dim(z) \) and \( \dim(v) \geq \dim(z) \). Ekeland (2006) relaxes this.

For simplicity, assume that \( \dim(\theta) = \dim(z) = \dim(v) \). Ekeland (2006) and Chiappori, McCann and Nesheim (2009) relax this.

From the FOC for the firm: \( v = v(z, P_z, B) \).

From the FOC for the consumer: \( \theta = \theta(z, P_z, P(z) + R, A) \).
Demand density:

\[ f_v(v(z, P_z, B)) \det \left[ \frac{\partial v(z, P_z, B)}{\partial z} \right] dv \]

Supply density:

\[ f_\theta(\theta(z, P_z, P(z) + R, A)) \det \left[ \frac{\partial \theta(z, P_z, P(z) + R, A)}{\partial z} \right] d\theta \]
Equilibrium in hedonic markets:
Supply = Demand at each point of $z$:

$$f_v(v(z, P_z, B)) \det \left[ \frac{\partial v(z, P_z, B)}{\partial z} \right] dz$$

$$= f_\theta(\theta(z, P_z, P(z) + R, A)) \det \left[ \frac{\partial \theta(z, P_z, P(z) + R, A)}{\partial z} \right] dz$$

This generates the equation for the market clearing wage function.
A Linear-Quadratic Example (Tinbergen, 1956)

- \( U(c, z, \theta, A) = R + P(z) + \theta'z - \frac{1}{2}z'Az \)
- \( \theta \) varies among agents but \( A \) does not.
- Consumer maximum:
  - FOC: \( \theta - Az + P_z = 0 \)
  - SOC: \( (P_{zz'} - A) \) is negative definite.
- Firm side: \( \Pi(z, v, B, P(z)) = v'z - \frac{1}{2}z'Bz - P(z) \)
  - FOC: \( v - Bz - P_z = 0 \)
  - SOC: \( -(B + P_{zz'}) \) is negative definite.
- \( \theta \sim \mathcal{N}(\mu_\theta; \Sigma_\theta) \) and \( v \sim \mathcal{N}(\mu_v; \Sigma_v) \).
One can show that the solution for the price function is:

\[ P(z) = \pi_0 + \pi_1' z + \frac{1}{2} z' \pi_2 z \]

(See Tinbergen, 1956).

Firm: \( v - Bz - \pi_1 - \pi_2 z = 0 \)

Consumer: \( \theta - Az + \pi_1 + \pi_2 z = 0 \)

Observe that demand and supply functions affine in \( \theta \) and \( v \), respectively.
\[ B + \pi_2 \]
\[ A - \pi_2 \]

positive definite

- From the first-order conditions, we have

\[ z = (B + \pi_2)^{-1}(\nu - \pi_1) \]
\[ z = (A - \pi_2)^{-1}(\theta - \pi_1) \]

- Hence,

\[ (B + \pi_2)^{-1}(\nu - \pi_1) = (A - \pi_2)^{-1}(\theta - \pi_1) \]
Equilibrium:

- Need demand equal to supply at each point of the quality spectrum.

\[
v = \pi_1 + (B + \pi_2)(A - \pi_2)^{-1}(\theta + \pi_1)
\]  

(6)

- Average Demand

\[
(B + \pi_2)^{-1}E(v - \pi_1) = E^D(z)
\]

- Average Supply

\[
(A - \pi_2)^{-1}E(\theta + \pi_1) = E^S(z)
\]
• Equality of Means

\[ \mu_\theta = E(\theta) \quad \mu_v = E(v) \]

Demand = Supply

\[ E^D(z) = E^S(z) \]

\[ (B + \pi_2)^{-1}(\mu_v - \pi_1) = (A - \pi_2)^{-1}(\mu_\theta + \pi_1) \]

\[ [(A - \pi_2)^{-1} + (B + \pi_2)^{-1}]^{-1} [(B + \pi_2)^{-1}\mu_v - (A - \pi_2)^{-1}\mu_\theta] = \pi_1 \]
**Equality of Variances**

**Demand**

\[
\Sigma_v = (B + \pi_2) \Sigma_z (B + \pi_2)'
\]

**Supply**

\[
\Sigma_{\theta} = (A - \pi_2) \Sigma_z (A - \pi_2)'
\]

Equate Demand to Supply:

\[
(B + \pi_2)^{-1} \Sigma_v (B + \pi_2)^{-1} = (A - \pi_2)^{-1} \Sigma_{\theta} (A - \pi_2)^{-1}
\]
- Initial conditions: \( U \geq \bar{U} \)
- Profits are positive: \( \Pi \geq 0 \)
- Nonnegativity of profits implies \(-\pi_0 \geq 0\)
- Similar argument on worker side
  - \( \pi_0 \geq 0 \)
- \( \therefore \pi_0 = 0. \)
Solution for Scalar Case:

Recall \( A - \pi_2 > 0, \ B + \pi_2 > 0. \)

\[
\pi_1 = \frac{\mu_v}{B + \pi_2} - \frac{\mu_\theta}{A - \pi_2}
+ \frac{1}{A - \pi_2} + \frac{1}{B + \pi_2}
\]

\[= \frac{\mu_v(A - \pi_2) - \mu_\theta(B + \pi_2)}{(B + \pi_2) + (A - \pi_2)}\]

\[= \mu_v \left( \frac{A - \pi_2}{A + B} \right) - \mu_\theta \left( \frac{B + \pi_2}{A + B} \right)\]
Equality of Variances

\[(A - \pi_2)^2 \sigma_{\nu\nu} = (B + \pi_2)^2 \sigma_{\theta\theta}\]

\[\pm (A - \pi_2) \sigma_{\nu} = (B + \pi_2) \sigma_{\theta}\]

\[\pi_2 = \frac{A \sigma_{\nu} - B \sigma_{\theta}}{\sigma_{\nu} + \sigma_{\theta}}\]

(other root inconsistent with second-order conditions)

\(\sigma_{\nu} > \sigma_{\theta}\) implies a bigger upper tail for demand for labor by firm. (Lots of very productive, upper tail firms)

This can be offset by greater concavity of worker preferences \(A < B\).
If $\sigma_v = \sigma_\theta$, $A = B$, $\pi_2 = 0$ is a solution.
If

\[ \sigma_v = \sigma_\theta, \quad A \neq B \]

\[ \frac{(A - B)}{2} = \pi_2 \]
Extract from “On the Theory of Income Distribution” (Tinbergen, 1956)

Econ 350, Winter 2010
James Heckman
In order to simplify our calculations we choose the units for $t_1$, and $t_2$ so as to make $\tau_1 = \tau_2 = 1$. Since $s_1$ and $s_2$ are expressed in the same units as $t_1$ and $t_2$ this may also be interpreted by saying that our symbols $s_1$, $s_2$, $s_1$, $s_2$, $t_1$, $t_2$, $t_1$, $t_2$ and $\sigma_1$, $\sigma_2$ represent the ratios of each of these to the corresponding $\tau_1$. 
The essence of the process of income formation may now be reformulated by saying that an income scale brings about a choice by each individual with regard to his "job" resulting in a correspondence between s-sets and t-sets illustrated by (5.3) (5.4) and that this correspondence will have to be such as to transform the t-distribution into an s-distribution identical with the given demand distribution. This identity is another expression for equality, in each compartment, of demand and supply. The procedure may be illustrated geometrically. Graph 1 shows the frequency distribution of \( t_1 \) and \( t_2 \), i.e. the supply side. On the frequency surface some curves of constant frequency densities \( f_1, f_2, f_3 \) have been drawn. Graph 2 shows the corresponding distribution of \( s_1 \) and \( s_2 \). Our income scale has to be one that will make the individuals around \( t_1^*, t_2^* \) choose a job around \( s_1^*, s_2^* \), and so on for each point in the \( t_1 - t_2 \)-plane, in such a way that corresponding points in the s- and the t-plane have the same frequencies. This does not mean that the frequency densities — i.e. the distances of the points \( t_1^* t_2^* \) and \( s_1^* s_2^* \) to the horizontal plane — are equal, but that the product of this distance and the surface elements are equal, that is:

\[
n \ (t_1^* t_2^*) \ dt_1^* \ dt_2^* = m \ (s_1^* s_2^*) \ ds_1^* \ ds_2^*
\]
The surface elements have been represented by the small rectangles drawn in graphs 3 and 4, where the projections of the curves of constant density on the horizontal plane have been drawn. These are ellipses, with the axes parallel to the coordinate axes when there is no correlation between the attributes, and an arbitrary position of the axes when there is correlation. The case represented by the graphs is characterized by \( r_t = 0 \) and \( r_s > 0 \).
The $t$-surface has to be "deformed" so as to coincide with the $s$-surface, otherwise there will not be equilibrium in all compartments of the market.
The process may now be shown algebraically. As is usual in mathematical problems of this type, we start by making an assumption about the shape of the function $\lambda$. We assume it to be of the form (5.5), where still the numerical values of the coefficients $\lambda_{00} \ldots \lambda_{02}$ are open. We try to find values for these coefficients which "do the job." If we are not able to find such values, our assumption (5.5) has to be abandoned. If it can be shown that there is only one solution, and our assumption appears to be a solution, then it will be the solution.
Identifying and Estimating the Model

- Two-step estimation procedure
- Estimate $P(z)$ from market data
- Use first-order conditions (1) and (3) in conjunction with the marginal prices obtained from step 1 to recover preferences and technology, respectively.
\[ \hat{\pi}_1 + \hat{\pi}_2 z = \mu_v - Bz + \varepsilon_v \]  

\[ \varepsilon_v = v - \mu_v \]  

\[ \hat{\pi}_1 + \hat{\pi}_2 z = -\mu_\theta + Az + \varepsilon_\theta \]  

\[ \varepsilon_\theta = \theta - \mu_\theta \]  

Estimated \( B \) and \( A \) are \( \hat{\pi}_2 \)! (not a satisfactory general solution)

But notice in scalar case if \( \sigma_v = 0, \pi_2 = -B \) is the correct answer.

If \( \sigma_\theta = 0, \pi_2 = A \) is the correct answer

but in general \( A \neq -B \).
Arguments Against Rosen’s Two Stage Procedure:

\[ \hat{P}_z(z) = \hat{\pi}_1 + \hat{\pi}_2 z \]

If the marginal valuation functions are nonlinear, the previous argument breaks down.

**Point One: Identification can only be obtained through arbitrary functional form assumptions**
Point Two: Absence of instruments

\[
\varepsilon_v = \varepsilon_\theta + (A - B)z + \mu_\theta(x) - \mu_v(y)
\]

(z, x, and y are functionally related to \(\varepsilon_v - \varepsilon_\theta\).)

Point Three: Need to use multimarket data

Need multimarket data:

See Heckman, Matzkin and Nesheim (2010).
Using All of the Economics of the Model

- Perturb the scalar version of the model.

- Profits are
  \[ \Pi(z) = v_0 + v_1z - \frac{b}{2}z^2 - P(z) \]
  \[ v_1 - bz - P'(z) = 0 \]
Worker preferences

\[ U(z) = \theta_0 + \theta_1 z - \frac{a}{2} z^2 + P(z) \]

\[ \theta_1 - az + P'(z) = 0 \]

- Nonlinearity is generic.
- Linearity is arbitrary and misleading.
Model 1, unrestricted $z$.

Suppose that $\nu$ and $\theta$ are non-normal (say 2-component mixture of normals)

<table>
<thead>
<tr>
<th>Components</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_{\nu_1}$</td>
<td>-1.0</td>
<td>0.0</td>
</tr>
<tr>
<td>$\mu_{\theta_1}$</td>
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<td>0.0</td>
</tr>
<tr>
<td>$\sigma^2_{\nu_1}$</td>
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<td>0.3</td>
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<tr>
<td>$\sigma^2_{\theta_1}$</td>
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<td>0.3</td>
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</table>
Model 2, linear-quadratic technologies, nonnegative $z$.

<table>
<thead>
<tr>
<th>Firms</th>
<th>$\Pi (z) = \nu_0 + \nu_1 z - \frac{1}{2}bz^2 - p (z)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\nu_1, b \geq 0$</td>
</tr>
<tr>
<td></td>
<td>$\ln \nu_1 = \nu_{10} + \nu_{11}' x + \epsilon_1$</td>
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<tr>
<td></td>
<td>$x$ and $\epsilon_1$ are both distributed as a mixture of normals</td>
</tr>
<tr>
<td></td>
<td>(the mixtures could have only one component).</td>
</tr>
<tr>
<td>FOC</td>
<td>$\nu_1 - bz - p' (z) = 0$</td>
</tr>
<tr>
<td>SOC</td>
<td>$-b - p'' (z) &lt; 0$</td>
</tr>
<tr>
<td>Workers</td>
<td>$V (z) = \theta_0 + \theta_1 z - \frac{1}{2}az^2 + p (z)$</td>
</tr>
<tr>
<td></td>
<td>$\theta_1, a \geq 0$</td>
</tr>
<tr>
<td></td>
<td>$\ln \theta_1 = \theta_{10} + \theta_{11}' y + \epsilon_2$</td>
</tr>
<tr>
<td></td>
<td>$y$ and $\epsilon_2$ are both distributed as mixtures of normals</td>
</tr>
</tbody>
</table>
Model 2.

<table>
<thead>
<tr>
<th>Components</th>
<th>1</th>
<th>2</th>
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<tbody>
<tr>
<td>$\mu_\nu$</td>
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<tr>
<td>$\mu_\theta$</td>
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</tr>
<tr>
<td>$\sigma^2_\nu$</td>
<td>0.21</td>
<td>0.21</td>
</tr>
<tr>
<td>$\sigma^2_\theta$</td>
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<td>0.61</td>
</tr>
<tr>
<td>$\lambda_\nu$</td>
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<td>0.5</td>
</tr>
<tr>
<td>$\lambda_\theta$</td>
<td>0.5</td>
<td>0.5</td>
</tr>
</tbody>
</table>
Slope of the price function, model 1a.
Curvature of the price function, model 1a.
Slope of the price function, model 1b.
Curvature of the price function, model 1b.

Figure 4: Curvature of Price Function - Model 1b

\[ P''(z) \]

\( \lambda = 0.99 \)

\( \lambda = 0.90 \)
Slope of the price function, model 2.
Curvature of the price function, model 2.
<table>
<thead>
<tr>
<th>Components</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_\nu$</td>
<td>0.0</td>
<td>1.0</td>
</tr>
<tr>
<td>$\mu_\theta$</td>
<td>1.0</td>
<td>-1.0</td>
</tr>
<tr>
<td>$\sigma^2_\nu$</td>
<td>0.5</td>
<td>1.0</td>
</tr>
<tr>
<td>$\sigma^2_\theta$</td>
<td>1.0</td>
<td>0.1</td>
</tr>
</tbody>
</table>
Slope of the price function, model 1.
Slope of the price function, model 1.

Figure 2: Curvature of Price Function: Model 1

-3 -2 -1 0 1 2 3
-1.5
-1
-0.5
0
0.5
1

\[ p''(z) \]

\[ \lambda = 0.5 \]
\[ \lambda = 0.9 \]
\[ \lambda = 1 \]

Figure 2: Curvature of Price Function: Model 1

-3 -2 -1 0 1 2 3
0
0.2
0.4
0.6
0.8
1

Density of z

\[ \lambda = 0.5 \]
\[ \lambda = 0.9 \]
\[ \lambda = 1 \]
Slope of the price function, model 2.
Elasticity of slope of price function with respect to $z$. 

\[ \lambda = 0.5 \]
\[ \lambda = 0.9 \]
\[ \lambda = 1 \]
Appendix
Bunching

- Classical hedonic model assumes equilibrium sorting of agents is smooth.
- No location has positive mass of people.
- Equilibrium pricing function is $C^2$.
- Are there conditions on primitives that rule out bunching?
SOC of Consumer

- For every consumer who chooses $z$, it must be the case that $U_{zz}(z, x, \theta) > P_{zz}(z)$.

$$\int \left( \frac{\Gamma_{zz}}{\Gamma_{zv}} \right) f_v f_y dy - \int \left( \frac{U_{zz}}{U_{z\theta}} \right) f_\theta f_x dx < \frac{\left( \int \frac{f_v f_y}{\Gamma_{zv}} dy - \int \frac{f_\theta f_x}{U_{z\theta}} dx \right)}{\int \tilde{Y} f_v f_y dy - \int \tilde{X} f_\theta f_x dx} < \sum_{i} \mathbb{E} \left( s_i(z, x) \right)

- For every firm who optimally chooses $z$, it must be the case that

$$\Gamma_{zz}(z, y, v) < P_{zz}(z).$$
SOC of Firm

$$\Gamma_{zz} \left( z, y, \tilde{d}(z, y) \right) < \frac{\int \left( \frac{\Gamma_{zz}}{\Gamma_{zv}} \right) f_v f_y dy - \int \left( \frac{U_{zz}}{U_{z\theta}} \right) f_\theta f_x dx}{\left( \int \frac{f_v f_y}{\Gamma_{zv}} dy - \int \frac{f_\theta f_x}{U_{z\theta}} dx \right)}$$

$$\Gamma_{zz} \left( z, y, \tilde{d}(z, y) \right) < U_{zz} \left( z, x, \tilde{s}(z, x) \right)$$
SOC of Firm

- Conditions depend in complicated way on curvatures of preferences and technology and on distribution of preferences and technology.

- In the special case of an additive FOC,

\[ \Gamma_{zz}(z) < U_{zz}(z). \]
No Bunching Example

- **Consumer problem:**
  \[
  \text{Max}_z \ P(z) - \frac{z^\beta}{\theta} \\
  \theta_l \leq \theta \leq \theta_u
  \]

- **Firm problem:**
  \[
  \text{Max}_z \ z^\alpha v - P(z) \\
  v_l \leq v \leq v_u
  \]
No Bunching Example

- FOC and SOC for the consumer’s problem:

  \[ FOC : P_z - \frac{\beta z^{\beta-1}}{\theta} = 0 \]

  \[ SOC : P_{zz} - \frac{\beta (\beta - 1) z^{\beta-2}}{\theta} < 0 \]

- FOC and SOC for the firm’s problem:

  \[ FOC : \alpha z^{\alpha-1} v - P_z = 0 \]

  \[ SOC : \alpha (\alpha - 1) z^{\alpha-2} v - P_{zz} < 0 \]
No Bunching Example

- Inverse supply and demand:

\[ \theta = \tilde{s}(z) = \frac{\beta z^{\beta-1}}{P_z} \quad \text{and} \quad v = \tilde{d}(z) = \frac{P_z}{\alpha z^{\alpha-1}}. \]

- Equilibrium condition:

\[
F_\theta \left( \frac{\beta z^{\beta-1}}{P_z(z)} \right) = F_v \left( \frac{P_z(z)}{\alpha z^{\alpha-1}} \right)
\]

\[
\frac{\beta z^{\beta-1}}{P_z(z)} = \frac{P_z(z)}{\alpha z^{1-\alpha}},
\]

for \( \theta_l \leq \frac{\beta z^{\beta-1}}{P'(z)} \leq \theta_u \).
No Bunching Example

- Equilibrium price function:

\[ P_z(z) = \left( \alpha \beta \cdot z^{\alpha+\beta-2} \right)^{1/2}, \]

where

\[ \theta_l^{\frac{2}{\beta-\alpha}} \left( \frac{\alpha}{\beta} \right)^{\frac{1}{\beta-\alpha}} \leq z \leq \theta_u^{\frac{2}{\beta-\alpha}} \left( \frac{\alpha}{\beta} \right)^{\frac{1}{\beta-\alpha}}. \]
No Bunching Example

- Supply function:

\[ z = \left( \frac{\alpha}{\beta} \right)^{\frac{1}{\beta - \alpha}} \theta^{\frac{2}{\beta - \alpha}}, \]

for \( \theta_l \leq \theta \leq \theta_u \).

- Demand function:

\[ z = \left( \frac{\alpha}{\beta} \right)^{\frac{1}{\beta - \alpha}} v^{\frac{2}{\beta - \alpha}}, \]

for \( v_l \leq v \leq v_u \) and \( \alpha < \beta \).
An Example of Equilibrium with Bunching

\[ \Gamma(z, \nu) = z^\alpha \nu, \text{ where } \alpha = .5 \text{ and } \nu \sim U(0, 1). \]

\[ \max_z z^\alpha \nu - P(z). \]

\[ \text{FOC: } \alpha z^{\alpha - 1} \nu - P_z(z) = 0. \]

\[ \text{SOC: } \alpha(\alpha - 1)z^{\alpha - 2} \nu - P_{zz}(z) < 0 \]
An Example of Equilibrium with Bunching

- FOC implies
  \[ v(z) = \frac{P_z(z) \, z^{1-\alpha}}{\alpha}. \]

- Consumer has a disutility of \( z \) given by
  \[ V(z, \theta) = z^\theta, \]
  where \( \theta \) is a random variable distributed \( U(.25, .75) \).
An Example of Equilibrium with Bunching

- \( \max_z P(z) - V(z, \theta) \).

- FOC: \( P_z(z) - \theta \, z^{\theta - 1} = 0 \).

- SOC: \( P_{zz}(z) - \theta (\theta - 1) \, z^{\theta - 2} < 0 \).

- \( \Gamma_{zz}(z, v(z)) < V_{zz}(z, \theta(z)) \)

- \( \Gamma_{zz}(z(v), v) < V_{zz}(z(\theta), \theta) \).

- \( z(v) = z(\theta) = z \).
An Example of Equilibrium with Bunching

- SOC are satisfied if and only if
  \[ \alpha(\alpha - 1)z^{\alpha-2}v < \theta(\theta - 1)z^{\theta-2}. \]

- The last condition becomes:
  
  (1) \( (\alpha - 1)z^{-1}P_z(z) < (\theta - 1)z^{-1}P_z(z) \)
  
  (2) \( \alpha < \theta \)

- \( z = \left(1 - \frac{\alpha}{2\theta}\right)^{\frac{1}{\theta-\alpha}} \)
An Example of Equilibrium with Bunching

- $v$ satisfies
  \[ v = \frac{\theta}{\alpha} \left( 1 - \frac{\alpha}{2\theta} \right) \]

- Firms with $v < .5$ and $\theta < .5$ will locate at $z = 0$. 
Linear-Quadratic Example (Tinbergen, 1956)

**Consumer Side with Vector** \( z \)

- Preferences quadratic in \( z \) and linear in \( P(z) \) (ignore \( x \)):
  \[
  U(c, z, \theta, A) = P(z) + \theta'z - \frac{1}{2}z'Az
  \]

- **FOC:**
  \[
  \theta - Az + P_z = 0 \quad (9)
  \]

- **SOC:**
  \[
  (P_{zz'} - A) \text{ is negative definite.}
  \]

- **Worker heterogeneity:** \( \theta \sim N(\mu_\theta, \Sigma_\theta) \).
Linear-Quadratic Example (Tinbergen, 1956)

**Firm Side**

- Production quadratic in $z$:

\[
\Pi(z, v, B, P(z)) = v'z - \frac{1}{2}z'Bz - P(z)
\]  

(10)

- FOC:

\[
v - Bz - P_z = 0
\]  

(11)

- SOC:

\[-(B + P_{zz'}) \text{ is negative definite.}\]

- Firm heterogeneity: $v \sim N(\mu_v, \Sigma_v)$. 
Equilibrium

- Equilibrium must satisfy (3).

- Given special structure, one can guess (correctly) that

\[ P(z) = \pi_0 + \pi_1 z + \frac{1}{2} z' \pi_2 z. \]

- Then check that the guess is correct.

- Firm FOC: \( v - Bz - \pi_1 - \pi_2 z = 0. \)

- Consumer FOC: \( \theta - Az + \pi_1 + \pi_2 z = 0. \)
Sorting Conditions

\[ z_D = (B + \pi_2)^{-1} (v - \pi_1) \]

\[ z_S = (A - \pi_2)^{-1} (\theta + \pi_1) \]

- Equate average demand to average supply:

\[ E^D(z) = (B + \pi_2)^{-1} E(v - \pi_1) \]

\[ E^S(z) = (A - \pi_2)^{-1} E(\theta + \pi_1) \]

- One vector equation in unknown coefficients:

\[ (B + \pi_2)^{-1} (\mu_v - \pi_1) = (A - \pi_2)^{-1} (\mu_\theta + \pi_1) \]
Sorting Conditions

\[ V^D (z) = \left( B + \pi_2 \right)^{-1} \Sigma_v \left( B + \pi_2 \right)^{-1'} \]

\[ V^S (z) = \left( A - \pi_2 \right)^{-1} \Sigma_\theta \left( A - \pi_2 \right)^{-1'} \]

• Second matrix equation:

\[ \left( A - \pi_2 \right)^{-1} \Sigma_\theta \left( A - \pi_2 \right)^{-1'} = \left( B + \pi_2 \right)^{-1} \Sigma_v \left( B + \pi_2 \right)^{-1'} \]

• Initial conditions:

\[ U(z) \geq U_0 \quad \text{and} \quad \Pi(z) \geq 0 \]
Sorting Conditions

- This implies $\pi_0 = 0$.

- Solution depends on

  1. Production and preference parameters $A, B$.
  
  2. Heterogeneity $\mu_v, \mu_\theta, \Sigma_v$, and $\Sigma_\theta$. 
Sorting Conditions

- Except in polar cases, price function does not directly reveal any individual structural parameters.

- Note that equilibrium matching implies

\[(B + \pi_2)^{-1}(v - \pi_1) = (A - \pi_2)^{-1}(\theta + \pi_1).\]  

- Functional and statistical dependence between \(v\) and \(\theta\).
Identifying and Estimating the Tinbergen Model

- Widely used two step estimate procedure (Rosen):

  1. Estimate $P(z)$ from market data.

  2. Use first-order conditions (9) and (11) in conjunction with the marginal prices obtained from step 1 to recover preferences and technology respectively.

- $\hat{\pi}_1$ and $\hat{\pi}_2$ are fitted price coefficients.
Identifying and Estimating the Tinbergen Model

- $x$ and $y$ are observable worker and firm regressors:

$$\hat{\pi}_1 + \hat{\pi}_2 z = \mu_v (y) - Bz + \omega_v \quad (13)$$

$$\hat{\pi}_1 + \hat{\pi}_2 z = -\mu_\theta (x) + Az - \omega_\theta \quad (14)$$

- $\omega_v$ and $\omega_\theta$ are unobservable:

$$\omega_v = v - \mu_v (y)$$

$$\omega_\theta = \theta - \mu_\theta (x)$$
Claim One: Identification Can Only be Obtained Through Arbitrary Functional Form Assumptions

- If fitted $P(z)$ is quadratic, linear functions of $z$ on left and right sides.

- If fitted $P(z)$ is not quadratic, nonlinearity can help with identification.

- This nonlinearity is arbitrary.
Claim One: Identification Can Only be Obtained Through Arbitrary Functional Form Assumptions

- However, small perturbations of above model lead to non-quadratic $P(z)$.

- Economics of the problem suggests that for many applications this quadratic equilibrium model not very good any way we want to move away from Tinbergen specification.

- Analysis of equilibrium equation shows that in fact, nonquadratic $P(z)$ is generic.
Example 1

- Perturb scalar version of quadratic model.

- Suppose heterogeneity distributed as mixture of normals with weights \((0.999, 0.001)\) or \((0.99, 0.01)\).

- Marginal price function is clearly nonlinear.

- Unattractive features of quadratic model:
  
  1. Negative and positive \(z\).
  
  2. Negative marginal products.
Example 2

- Restrict marginal product to be positive.
- Restrict marginal utility of work to be negative.
- The non-identified case is uninteresting as well as unlikely.
Claim Two: Endogeneity Problem

- $z$ is endogenous in (13) and (14).

- Rewrite (12) as

$$\omega_v = \omega_\theta + (A - B)z + \mu_\theta(x) - \mu_v(y).$$

- Conditional on $z$, there is a functional and statistical dependence between $x$, $y$, $\omega_v$, and $\omega_\theta$. 
Claim Three: Use of Multimarket Data

- Provides variation not available in a single cross section.

- But why do prices vary across markets?

- Must specify which elements vary across markets, and which elements do not.

- Our results show there is no need for multimarket data.
Generic Identification of General Additive Scalar Model

- Establish points made in previous section more formally and more generally:

  2. Recover all structural parameters up to location.

- Results apply to any model in which FOC reduce to additive first-order conditions.

- Generalization of quadratic model.

- Can constrain to have economically meaningful interpretations.
## Model 1, Unrestricted $z$

<table>
<thead>
<tr>
<th>Components</th>
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</thead>
<tbody>
<tr>
<td>$\mu_{\nu_1}$</td>
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<tr>
<td>$\mu_{\theta_1}$</td>
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<td>0.0</td>
</tr>
<tr>
<td>$\sigma^2_{\nu_1}$</td>
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<td>0.3</td>
</tr>
<tr>
<td>$\sigma^2_{\theta_1}$</td>
<td>0.5</td>
<td>0.3</td>
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## Table 2

### Model 2

### Linear Quadratic Technologies

### Non-Negative $z$

<table>
<thead>
<tr>
<th>Category</th>
<th>Equation</th>
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<tbody>
<tr>
<td>Firms</td>
<td>$\Pi(z) = \nu_0 + \nu_1 z - \frac{1}{2} b z^2 - p(z)$</td>
</tr>
<tr>
<td></td>
<td>$\nu_1, b \geq 0$</td>
</tr>
<tr>
<td></td>
<td>$\ln \nu_1 = \nu_{10} + \nu_{11}' x + \varepsilon_1$</td>
</tr>
<tr>
<td></td>
<td>$x$ and $\varepsilon_1$ are both distributed as a mixture of normals</td>
</tr>
<tr>
<td></td>
<td>(the mixtures could have only one component)</td>
</tr>
<tr>
<td>FOC</td>
<td>$\nu_1 - b z - p'(z) = 0$</td>
</tr>
<tr>
<td>SOC</td>
<td>$-b - p''(z) &lt; 0$</td>
</tr>
<tr>
<td>Workers</td>
<td>$V(z) = \theta_0 + \theta_1 z - \frac{1}{2} a z^2 + p(z)$</td>
</tr>
<tr>
<td></td>
<td>$\theta_1, a \geq 0$</td>
</tr>
<tr>
<td></td>
<td>$\ln \theta_1 = \theta_{10} + \theta_{11}' y + \varepsilon_2$</td>
</tr>
<tr>
<td></td>
<td>$y$ and $\varepsilon_2$ are both distributed as mixtures of normals</td>
</tr>
<tr>
<td>Components</td>
<td>1</td>
</tr>
<tr>
<td>------------</td>
<td>---</td>
</tr>
<tr>
<td>$\mu_\nu$</td>
<td>1.0</td>
</tr>
<tr>
<td>$\mu_\theta$</td>
<td>-0.5</td>
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<td>$\sigma^2_\nu$</td>
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<td>$\sigma^2_\theta$</td>
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</tr>
<tr>
<td>$\lambda_\nu$</td>
<td>0.5</td>
</tr>
<tr>
<td>$\lambda_\theta$</td>
<td>0.5</td>
</tr>
</tbody>
</table>
Figure 1: Slope of Price Function - Model 1a
Figure 2: Curvature of Price Function - Model 1a

The diagram shows the curvature of the price function for Model 1a. The graph plots $P''(z)$ against $z$, with two lines indicating different values of $\lambda$: $\lambda = 1.00$ (solid line) and $\lambda = 0.999$ (dashed line).
Figure 3: Slope of Price Function - Model 1b

\[ P''(z) \]

- \( \lambda = 0.99 \)
- \( \lambda = 0.90 \)
Figure 4: Curvature of Price Function - Model 1b
Figure 5: Slope of Pricing Function - Model 2
Figure 6: Curvature of Price Function - Model 2
<table>
<thead>
<tr>
<th>Components</th>
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<th>2</th>
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</thead>
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<tr>
<td>$\mu_\nu$</td>
<td>0.0</td>
<td>1.0</td>
</tr>
<tr>
<td>$\mu_\theta$</td>
<td>1.0</td>
<td>-1.0</td>
</tr>
<tr>
<td>$\sigma^2_\nu$</td>
<td>0.5</td>
<td>1.0</td>
</tr>
<tr>
<td>$\sigma^2_\theta$</td>
<td>1.0</td>
<td>0.1</td>
</tr>
</tbody>
</table>
Figure 1: Slope of Price Function: Model 1

\[ p'(z) \]

\[ \lambda = 0.5 \]
\[ \lambda = 0.9 \]
\[ \lambda = 1 \]
Figure 2: Curvature of Price Function: Model 1
Figure 3: Slope of Price Function: Model 2

\[ p'(z) \]

\[ \lambda = 0.5 \]
\[ \lambda = 0.9 \]
\[ \lambda = 1 \]
Figure 4: Elasticity of Slope of Price Function with Respect to $z$

---

The figure shows the elasticity of slope of the price function with respect to $z$. The x-axis represents $z$ with values ranging from 0 to 5, while the y-axis represents elasticity with values ranging from -1.5 to 1.5. The graph includes lines for different values of $\lambda$: $\lambda = 0.5$, $\lambda = 0.9$, and $\lambda = 1$. The density of $z$ is also plotted on a separate graph, with the x-axis representing $z$ and the y-axis representing the density.
Parametric and nonparametric analyses of a one-dimensional model with additively separable first order conditions

- Production technology $F(z, x, \varepsilon_1)$ where $(x, \varepsilon_1) = \nu$:

$$F_z(z, x, \varepsilon_1) = P'(z)$$
\[ \psi_1(F_z(z, x, \varepsilon)) = \tau(z) + M_1(\eta_1(x) + \varepsilon_1) \]

(a) \( \psi_1 \) the identity function and

\[ F_z(z, x, \varepsilon) = \varphi_1(z) + M_1(\eta_1(x) + \varepsilon_1), \]

where \( \frac{d\varphi_1}{dz} < 0; \) \( M_1 \) can be the identity function, the exponential function or any other monotonic transformation of \( \eta_1(x) + \varepsilon_1 \). Or . . .

(b) \( \psi_1(q) = \log(q) \)

\[ F_z(z, x, \varepsilon) = K_1(z)M_1(\eta_1(x) + \varepsilon_1), \]

\( M_1 \) monotonic

\[ \eta_1(x) + \varepsilon_1 = (M_1)^{-1} \left( \frac{P'(z)}{K_1(z)} \right) \]
Consider one member of this class:

\[ F(z, x, \varepsilon) = \Phi^1(z) + z\eta_1(x) + z\varepsilon_1 \]  \hspace{1cm} (A-2)

Profit maximization:

\[ \frac{\partial \Phi^1}{\partial z} = \varphi_1(z) \]

FOC: \( \varphi_1(z) = \eta_1(x) + \varepsilon_1 = P'(z) \)

SOC: \( \varphi'_1(z) - P''(z) < 0 \)
Consumer side:

\[
U(z, y, \varepsilon_2, c) = c - \Phi^2(z) + z\eta_2(y) + z\varepsilon_2
\]

Define \( \frac{\partial \Phi^2}{\partial z} = \varphi_2(z) \).

FOC: \( P'(z) - \varphi_2(z) + \eta_2(y) + \varepsilon_2 = 0 \); and,

SOC: \( P''(z) - \varphi'_2(z) < 0 \)

\[
\begin{cases}
\varepsilon_1 = P'(z) - \varphi_1(z) - \eta_1(x) \\
x = x
\end{cases}
\begin{cases}
\varepsilon_2 = P'(z) - \varphi_2(z) - \eta_2(y) \\
y = y
\end{cases}
\]
Equilibrium:

\[
\int_{\chi} g_1(P'(z) - \varphi_1(z) - \eta_1(u))(P''(z) - \varphi'_1(z))q_1 \, du
\]

\[
= \int_{\psi} g_2(P'(z) - \varphi_2(z) - \eta_2(u))(P''(z) - \varphi'_2(z))q_2 \, du
\]
**Definition (Genericity)**

Property $P(\theta)$, depending on a parameter $\theta \in \Theta$, is called generic if the set $\Omega \subset \Theta$ of values of the parameter for which it holds true contains a countable intersection of open dense subsets. If $\Theta$ is a complete metric space, such a set $\Omega$ will be dense in $\Theta$, by a celebrated theorem of Baire. Moreover, the intersection of two such sets will still be dense in $\Theta$. In other words, if a property is generic, and does not hold for a certain value $\bar{\theta}$ of the parameter, there will be in any neighborhood of $\bar{\theta}$ some other value $\theta$ of the parameter where the property holds true. A generic property is robust in the sense that if $P_1(\theta)$ and $P_2(\theta)$ are generic, then so is their intersection $P_1(\theta) \cap P_2(\theta)$.
Theorem (1)

Generically with respect to any of the parameter pairs, the equilibrium equations have no solution of the form

\[ P'(z) = a_1 + b_1 \varphi_1(z), \]  
\[ P'(z) = a_2 + b_2 \varphi_2(z). \]

- As a consequence, Brown-Rosen Point One that regressions of \( P'(z) \) on \( \varphi_1(z) \) or \( \varphi_2(z) \) simply recover the marginal price \( (\hat{a}_1 = 0, \hat{b}_1 = 1; \hat{a}_2 = 0, \hat{b}_2 = 1) \) is not generically correct.

- We can identify \( (g_1, g_2, \varphi_1, \varphi_2, \eta_1, \eta_2) \) from data on \( P(z), z, x \) and \( y \) from single market.
\[ F_{x_i}^1(z \mid x) = -g_1(T_1(z) - \eta_1(x)) \cdot \frac{\partial \eta_1}{\partial x_i} \quad (15) \]

\[ \frac{F_{x_i}^1(z \mid x)}{F_{x_j}^1(z \mid x)} = \frac{\partial \eta_1(x)}{\partial x_i} \cdot \frac{\partial \eta_1(x)}{\partial x_j} \quad \text{for all } i, j \quad (16) \]

\[ \left( \frac{-F_z^1(z \mid x)}{F_{x_i}^1(z \mid x)} \right) = \frac{T_1'(z)}{\frac{\partial \eta_1(x)}{\partial x_i}} \quad (17) \]

\[ \text{sign} \left( F_{x_i}^1 \right) = -\text{sign} \left( \frac{\partial \eta_i}{\partial x_i} \right) \quad \frac{\partial \eta_i}{\partial x_i} > 0 \]
\[
\frac{\partial}{\partial z} \log \left[ \frac{-F_{z}^{1}(z \mid x)}{F_{x_i}^{1}(z \mid x)} \right] = \frac{T_1''(z)}{T_1'(z)}
\]

\[
h(z, x) = \log \left[ \frac{-F_{z}^{1}(z \mid x)}{F_{x_i}^{1}(z \mid x)} \right]
\]

\[
h(z, x) = h_0 + h_1(z) + h_2(x)
\]

\[
\frac{d h_1(z)}{dz} = \frac{T_1''(z)}{T_1'(z)}
\]
\[ T_1'(z) = K_1 \exp h_1(z) \]  

(19)

\[ T_1(z) = C_1 + K_1 \int_0^z \exp(h_1(s)) \, ds \]

\[ \frac{\partial \eta_1}{\partial x_i} \exp (h_0 + h_1(z) + h_2(x)) = K_1 \exp (h_1(z)) \]

\[ \frac{\partial \eta_1(x)}{\partial x_i} = K_1 \exp (-h_0 - h_2(x)) \]  

(20)

\[ \eta_1(x) = R_1 + K_1 \int_0^x \exp (-h_0 - h_2(s)) \, ds \]
We can identify $T_1(z)$ and $\eta_1(x)$ up to constants

$$\tilde{\eta}_1(x) = \frac{\eta_1(x) - R_1}{K_1}$$

$$\tilde{T}_1(z) = \frac{T_1(z) - C_1}{K_1}$$

$$\varepsilon_1 = T_1(z) - \eta_1(x)
= (C_1 - R_1) + K_1(\tilde{T}(z) - \tilde{\eta}_1(x)).$$
Assume

(1) \(E(\varepsilon_1) = 0\), or

(2) \(\text{median}(\varepsilon_1) = 0\).

\(\tilde{\varepsilon}_1 = (\varepsilon_1 / K_1)\)

\[g_1(\varepsilon_1) \, d\varepsilon_1 = K_1 g_1(\tilde{\varepsilon}_1 K_1) \, d\tilde{\varepsilon}_1 = \tilde{g}_1(\tilde{\varepsilon}_1)\]
From $F^2(z \mid y)$ we can identify:

$$\tilde{\eta}_2(y) = \frac{\eta_2(y) - R_2}{K_2}$$

$$\tilde{T}_2(z) = \frac{T_2(z) - C_2}{K_2}$$

$$\tilde{g}_2(\tilde{\varepsilon}_2) = K_2g_2(K_2\varepsilon_2) \, d\varepsilon_2$$

$$\tilde{\varepsilon}_2 = \frac{\varepsilon_2}{K_2}$$

$$\tilde{\varphi}_2(z) = P'(z) + K_2\tilde{T}_2(z) + C_2$$
Lack of identification of the scale of the utility function is a classical result:

\[
F(z, x) = \Phi^1(z) + z\eta_1(x) + z\left(P'(z) + \eta_1(x) - \varphi_1(z)\right)
\]

\[
= \Phi^1(z) + zP'(z) - z\varphi_1(z)
\]

\[
\psi(z) = F(z, x) - zP'(z)
\]

\[
= \int_0^z \varphi_1(t) \, dt - z\varphi_1(z)
\]
\[
\frac{\partial \psi(z)}{\partial z} = -z \varphi_1'(z)
\]

\[
C_0 + \int \left[ -\frac{1}{z} \frac{\partial \psi(z)}{\partial z} \right] \, dz = \varphi_1(z)
\]

- We now assume that there is a finite-dimensional vector space \( E \) which contains both \( \varphi_1 \) and \( \varphi_2 \), and which is known \( \text{ex ante} \).

- Assume \( \varphi_1 \) and \( \varphi_2 \) can be described by a finite set of polynomials \((a_1, \ldots, a_K)\) and \((b_1, \ldots, b_K)\).
**Theorem (2)**

*Generically with respect to any of the parameter pairs in Theorem 1, no solution $P$ of the equilibrium equation belongs to $E$, and $\varphi_1$ and $\varphi_2$ are identified up to additive constants.*
Is there information in the joint densities?

**Theorem (3)**

The joint density provides no more information than the marginal densities \( f(z_1, x) \) and \( f(z_2, y) \).
**Instrumental Variables**

- \( P'(z) = \varphi_1(z) + \eta_1(x) + \varepsilon_1. \)

- \( z \in \mathcal{Z} = (0, \infty), \ x \in \mathcal{X} = (0, \infty), \) and \( \varepsilon_1 \in \mathcal{E}_1 = (0, \infty), \)
  where \( (x, \varepsilon) \sim q_1(x)g_1(\varepsilon). \)

- \( P'(z) > 0, \ \varphi_1(z) > 0, \) and \( P'' - \varphi'_1 > 0. \)

- We assume that \( \mathbb{E}_X(\eta^2_1(x)) < \infty. \)
Corollary (1) of Theorem (1)

Generically with respect to any pair of the parameters in Theorem 1, $E_z(\varphi_1(z) \mid x)$ cannot be collinear with $\eta_1$. 
Summary, Conclusions, and Proposed Extensions