SORTING AND FACTOR INTENSITY: PRODUCTION AND UNEMPLOYMENT ACROSS SKILLS

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Motivation

Background:
- Applied micro theory: two-sided heterogeneity under one-to-one
- Macro / Labor / Trade / Urban / Development: intensive margin

Trade-Off: better workers vs. more workers

Examples:
- assignment of managerial time ("span of control": Sattinger 1975, Lucas 1978,...)
- assignment of land, of "distance", of production...

Goals:
1. Capture factor intensity in tractable manner (no peer effects)
2. Characterize sorting condition: complementarity with quality vs. complementarity with quantity
3. Characterize factor intensity, assignment, unemployment, wages
4. Tie it in with frictional theories of hiring
**Motivation**

Resources / Firms

Workers

$h^f_1$

$h^f_2$

$h^w_1$

$h^w_2$
Motivation

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Workers

\( h_f^1 \)

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MOTIVATION

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- \( r_{ij} \) resources of firm type \( i \) devoted to worker type \( j \), \( r_{i1} + r_{i2} \leq h_i^f \)
- \( l_{ji} \) labor of worker type \( j \) deployed at firm type \( i \), \( l_{j1} + l_{j2} \leq h_j^w \)

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- Frictional Markets: one-on-one matching, but similar flavor under comp. search (Shimer-Smith 00, Atakan 06, Mortensen-Wright 03, Shi 02, Shimer 05, Eeckhout-Kircher 10)
The Model

- Population

- Production of firm y

- Preferences
THE MODEL

- **Population**
  - Workers of type $x \in X = [\underline{x}, \bar{x}]$, distribution $H^w(x)$
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- **Production of firm $y$**

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  - $F(x, y, l_x, r_x)$, where $l_x$ workers of type $x$, $r_x$ fraction of firm’s resources
  - $F$ increasing in all arguments
  - $F$ str. concave in each of the last two arguments
  - $F$ constant returns to scale in last two arguments
  - Total output of the firm: $\int F(x, y, l_x, r_x)dx$
  - Production with one worker type: $f(x, y, l) = F(x, y, l, 1)$

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Different resource levels: $F(x, y, l, r) = \tilde{F}(x, y, l, rT(y))$.
Generic capital: $F(x, y, l, r) = \max_k \tilde{F}(x, y, l, r, k) - ik$.
Competitive search: $F(x, y, l, r) = \max_v \tilde{F}(x, y, vm(l/v), r) - vc$.
The Model

Hedonic wage schedule $w(x)$ taken as given.

- **Optimization:**

- **Feasible Resource Allocation:**

- **Competitive Equilibrium**
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  - Firms maximize: $\max_{l_x, r_x} \int [F(x, y, l_x, r_x) - w(x)l_x]dx$
  - Implies: $r_x > 0$ only if $(x, \frac{lx}{rx}) = \arg \max f(x, y, \theta) - \theta w(x) \quad (*)$

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- **Feasible Resource Allocation:**
  - $\mathcal{R}(x, y, \theta)$: resources to any $x' \leq x$ by any $y' \leq y$ with $\frac{l_{x'}}{r_{x'}} \leq \theta$.
    1. Resource feasibility $[\mathcal{R}(y | X, \Theta) \leq H^f(y) \forall y]$
    2. Worker feasibility $[\int_{\theta \in \Theta} \int_{x' \leq x} \theta d\mathcal{R}(\theta, x' | Y) \leq H^w(x) \forall x]$

- **Competitive Equilibrium**
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Hedonic wage schedule \( w(x) \) taken as given.

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- **Competitive Equilibrium** is a tuple \((w, \mathcal{R})\) s.t.
  - 1 Optimality Cond. \( [(x, y, \theta) \in \text{supp} \mathcal{R} \text{ only if it satisfies } (*)] \)
  - 2 Market Clearing \( [\int \theta d\mathcal{R}(\theta|X, Y) \leq h^w(x), \ \text{"=} \text{ if } w(x) > 0, \forall x] \)
ASSORTATIVE MATCHING

**Definition (Assortative Matching)**

A resource allocation $\mathcal{R}$ entails positive (negative) sorting if its support only comprises points $(x, \mu(x), \theta(x))$ with $\mu'(x) > 0$ ($< 0$).

**Proposition (Condition for Assortative Matching)**

A necessary condition to have equilibria with positive assortative matching (PAM) is that $F_{12} \geq F_{23} F_{14}$ holds along the equilibrium path. If this holds everywhere with strict inequality, this suffices to ensure that only PAM equilibria exist (and that PAM is efficient). The reverse inequality has the same implications for negative assortative matching.

Next: interpretation, sketch of proof, graphical intuition, efficiency, examples, characterization of resource allocation.
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**SKETCH OF PROOF OF PAM-CONDITION**

Assume PAM allocation with resources on \((x, \mu(x), \theta(x))\). Must be optimal, i.e., maximizes:

\[
\max_{x,\theta} f(x, \mu(x), \theta) - \theta w(x).
\]

First order conditions:

\[
\begin{align*}
  f_\theta(x, \mu(x), \theta(x)) - w(x) &= 0 \quad (1) \\
  f_x(x, \mu(x), \theta(x)) - \theta(x) w'(x) &= 0, \quad (2)
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The Hessian is

\[
Hess = \begin{pmatrix}
 f_{\theta \theta} & f_{x \theta} - w'(x) \\
 f_{x \theta} - w'(x) & f_{xx} - \theta w''(x)
\end{pmatrix}.
\]

Second order condition requires \(|Hess| \geq 0\):

\[
f_{\theta \theta} [f_{xx} - \theta w''(x)] - (f_{x \theta} - w'(x))^2 \geq 0. \quad (3)
\]

Differentiate (1) and (2) with respect to \(x\), substitute:

\[
-\mu'(x) [f_{\theta \theta} f_{xy} - f_{y \theta} f_{x \theta} + f_{y \theta} f_x / \theta] \geq 0
\]

Positive sorting means \(\mu'(x) > 0\), requiring [...] < 0 and after rearranging:

\[
F_{12} F_{34} \geq F_{23} F_{14}. \quad (4)
\]
INTUITION FOR SORTING CONDITION

Budget Set: $D = \{(x, l) | lw(x) \leq M\}$

Iso-output Curve: $iy = \{(x, l) | F(x, y, l, 1) = \Pi\}$

Slope of Iso-output Curve:
$$\frac{\partial l}{\partial x} = -\frac{F_1(x, y, l, 1)}{F_3(x, y, l, 1)}.$$

Fix $F_{23} > 0$ and consider better firm:
- If $F_{12} \approx 0$, higher $y$ has flatter slope (numerator is constant).
- If $F_{12} \gg 0$, then higher $y$ will have steeper slope.
Intuition for Sorting Condition

Budget Set: \( D = \{(x, l)|lw(x) \leq M\} \)

Iso-output Curve: \( i_y = \{(x, l)|F(x, y, l, 1) = \Pi\} \)

Slope of Iso-output Curve: \( \frac{\partial l}{\partial x} = -\frac{F_1(x,y,l,1)}{F_3(x,y,l,1)} \).

Fix \( F_{23} > 0 \) and consider better firm:

- If \( F_{12} \approx 0 \), higher \( y \) has flatter slope (numerator is constant).
- If \( F_{12} \gg 0 \), then higher \( y \) will have steeper slope.
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Special Cases

Efficiency Units of Labor

- Skill "=" Quantity: $F(x, y, l, r) = \tilde{F}(y, xl, r) \Rightarrow F_{12}F_{34} = F_{23}F_{14}$

Multiplicative Separability

- $F(x, y, l, r) = A(x, y)B(l, r)$. Sorting: $[AA_{12}/(A_1A_2)][BB_{12}/(B_1B_2)] \geq 1$
- If $B$ is CES with substitution $\epsilon$: $[AA_{12}/(A_1A_2)] \geq \epsilon$. (root-sm; E-K 10)

Becker’s one-on-one matching

- $F(x, y, \min\{l, r\}, \min\{r, l\}) = F(x, y, 1, 1) \min\{l, r\}$
- Like inelastic CES ($\epsilon \rightarrow 0$), so sorting if $F_{12} \geq 0$

Sattinger’s span of control model

- $F(x, y, l, r) = \min\{\frac{r}{l(x, y)}, l\}$; written as CES between both arguments:
- Our condition converges for inelastic case to log-supermod. in qualities
**Special Cases**

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**Becker’s one-on-one matching**
- \( F(x, y, \min\{l, r\}, \min\{r, l\}) = F(x, y, 1, 1) \min\{l, r\}\),
- Like inelastic CES \( (\epsilon \to 0) \), so sorting if \( F_{12} \geq 0 \)

**Sattinger’s span of control model**
- \( F(x, y, l, r) = \min\{\frac{r}{l(x, y)}, l\} \); written as CES between both arguments:
  - Our condition converges for inelastic case to log-supermod. in qualities

**Extension of Lucas’ span of control model**
- \( F(x, y, l, r) = yg(x, l/r)r \), sorting only if good types work less well together \( (g_{12}/g_1 - g_{22}/g_2 \geq 1/\theta) \).

**Spatial sorting in mono-centric city:**
- \( F(x, y, l, r) = l(xg(y) + v(r/l)) \Rightarrow \) higher earners in center.
**Factor Intensity, Assignment, Wages**

**Proposition (Factor Intensity and Assignment)**

If the sorting condition holds, then the equilibrium assignment, factor intensities and wages are determined by the system of differential equations:

\[
\mu'(x) = \frac{h_w(x)}{\theta(x) h_f(x)}, \quad \theta'(x) = \frac{1}{f_{\theta \theta}} \left[ \frac{1}{\theta} f_x - \frac{h_w}{\theta h_f} f_y - f_{x \theta} \right], \quad w'(x) = \frac{f_x}{\theta}.
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Proof: $\mu'$ from market clearing: $H_w(\bar{x}) - H_w(x) = \int_{\mu(x)}^{\bar{y}} \theta(\bar{x})h_f(\bar{x}) dx$

$$\theta' \text{ from FOC: } f_{\theta} = w(x) \text{ and } f_x/\theta = w', \text{ diff. and subst. } \mu'.$$

Example: $F(x, y, l, r) = A(x, y)(\alpha l^\gamma + (1 - \alpha)r^\gamma)^{1/\gamma}$, uniform distr.

- symmetry $A$ and $\alpha = 1/2$: then "one-to-one" ($\theta(x) = 1$ and $\mu(x) = x$)
- symmetric $A$ but $\alpha < 1/2$: then $\theta' > 0$

Below lowest assigned worker type: Unemployment
ILLUSTRATION OF EXTENSIONS
GENERAL CAPITAL, MONOPOLISTIC COMPETITION
ILLUSTRATION OF EXTENSIONS

GENERAL CAPITAL, MONOPOLISTIC COMPETITION

General Capital:

- $F(x, y, l, r) = \max_k \hat{F}(x, y, l, r, k) - ik$ (CRS in quantities)
- sorting condition: $\hat{F}_{12} \hat{F}_{34} \hat{F}_{55} - \hat{F}_{12} \hat{F}_{35} \hat{F}_{45} - \hat{F}_{15} \hat{F}_{25} \hat{F}_{34} \geq \hat{F}_{14} \hat{F}_{23} \hat{F}_{55} - \hat{F}_{14} \hat{F}_{25} \hat{F}_{35} - \hat{F}_{15} \hat{F}_{23} \hat{F}_{45}$.

Monopolistic Competition in the Output Market:

- consumers have CES preferences with substitution $\rho$
- sales revenue of firm $y$: $\chi F(x, y, l, 1)^\rho$
- Sorting condition

$$\left[ \rho \tilde{F}_{12} + (1 - \rho)(\bar{F}) \frac{\partial^2 \ln \tilde{F}}{\partial x \partial y} \right] \left[ \rho \tilde{F}_{34} - (1 - \rho) l \tilde{F} \frac{\partial^2 \ln \tilde{F}}{\partial l^2} \right] \geq \left[ \rho \tilde{F}_{23} + (1 - \rho) \tilde{F} \frac{\partial^2 \ln \tilde{F}}{\partial y \partial l} \right] \left[ \rho \tilde{F}_{14} + (1 - \rho) \left( l \tilde{F} \frac{\partial^2 \ln \tilde{F}}{\partial x \partial r} \right) \right].$$

- independent of $\chi$
- our condition under $\rho = 1$, log-sm when production linear in $l$. 
ILLUSTRATION OF EXTENSIONS
SEARCH WITH LARGE FIRMS (COMPETITIVE SEARCH)

(Smith 99, Acemoglu-Hawkins 06, Mortensen 09, Helpman-Itsikhoki-Redding 10, Menzio-Moen 10,...).

Vacancy filling prob.: $m(q)$. Job finding prob.: $m(q)/q$. Posting ($x,v_x,\omega_x$).

$$\max_{r_x,l_x,\omega_x,v_x} \int [F(x,y,l_x,r_x) - l_x \omega_x - v_x c] \, dx$$

s.t. $l_x = v_x m(q_x); \quad \text{and} \quad \omega_x m(q_x)/q_x = w(x)$.

Two equivalent formulations:

1. $\max s_x, r_x \int [G(x,y,s_x,r_x) - w(x)s_x] \, dx$, where $G(x,y,s_x,r_x) = \max v_x [F(x,y,v_x m(s_x/v_x),r_x) - v_x c]$.

2. $\max r_x, l_x, v_x \int [F(x,y,l_x,r_x) - C(x,l_x)] \, dx$, where $C(x,l_x) = \min v_x, q_x c v_x + q_x v_x w(x)$.

From 1.: check sorting, compute $w(x)$ as in previous part.

From 2.: determine unemployment. FOC (simple closed form with const. elasticity $\alpha$):

$$w(x) q_x = \eta(q) \frac{1 - \eta(q)}{\eta(q)} c = 1 - \frac{\alpha}{\alpha} c$$

Proposition 3: The unemployment rate is falling in worker type.
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**CONCLUSION**

This work:
- Lay out a tractable sorting model with factor intensity
- Derive tractable sorting condition \((F_{12}F_{34} \geq F_{14}F_{23})\)
- Characterize equilibrium factor intensity, assignment and wages
- Extend to frictional market with sorting and large firms
- Various other extensions (general capital, monop. comp.)

Future:
- Generate more work on sorting on the intensive market
- Comparative statics on consequences of aggregate changes
- Applications in trade/macro/urban...
ILLUSTRATION OF EXTENSIONS

COMPETITIVE SEARCH WITH LARGE FIRMS

\[ \text{Wage posting and frictional hiring (Peters 90, Burdett-Shi-Wright 01, ...)} \]

- Searchers per vacancy: \( q = \frac{s}{v} \)
- Vacancy filling prob: \( m \)
- Job finding prob: \( \frac{m}{q} \)
Illustration of Extensions
Competitive Search with Large Firms

Wage posting and frictional hiring (Peters 90, Burdett-Shi-Wright 01,...)

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- Job finding prob: \( \frac{m}{q} = \frac{1}{2} \)
Wage posting and frictional hiring (Peters 90, Burdett-Shi-Wright 01,...)

- Searchers per vacancy: $q = \frac{s}{v}$
- Vacancy filling prob: $m$
- Job finding prob: $\frac{m}{q}$
ILLUSTRATION OF EXTENSIONS
COMPETITIVE SEARCH WITH LARGE FIRMS

Wage posting and frictional hiring (Peters 90, Burdett-Shi-Wright 01, ...)

- Searchers per vacancy: $q = s/v$
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- Job finding prob: $m/q \rightarrow \frac{1 - e^{-q}}{q}$
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ILLUSTRATION OF EXTENSIONS
COMPETITIVE SEARCH WITH LARGE FIRMS

Wage posting and frictional hiring (Peters 90, Burdett-Shi-Wright 01,...)

- Searchers per vacancy: $q = s/v$
- Vacancy filling prob: $m(q) \to 1 - e^{-q}$, $m'$
- Job finding prob: $m/q \to \frac{1-e^{-q}}{q}$, $\omega \frac{m(q')}{q'} = \omega \frac{m(q)}{q} = W(x)$