Linear IV and Simultaneous Equations

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Setup

- Linear regression model
  \[ Y = X\beta + \varepsilon \]  

- Endogeneity of \( X \) means that \( X \) and \( \varepsilon \) are correlated, i.e. \( E(X'\varepsilon) \neq 0 \).

- Suppose we observe another variable \( Z \) (the instrument) which is uncorrelated with \( \varepsilon \). We postulate that the relation between \( Z \) and \( X \) is of the form
  \[ X = Z\gamma + u \]  

- Equation (2) is known as the first stage.
Setup

*Note that we can combine equations (1) and (2) to form the reduced form:*

\[ Y = Z\pi + \eta \]  

where \( \pi = \gamma\beta \) and \( \eta = u\beta + \varepsilon \).

*Our goal: consistently estimate \( \beta \).*

*In all IV settings there will be two required conditions:*

- **Rank**: In this case: \( Z \) is correlated with \( X \). (testable)
- **Independence**: In this case: \( Z \) is uncorrelated with \( \varepsilon \). (not testable)

*Suppose throughout that \( \text{dim}(X) = n \times 1 \) so there is only one endogenous regressor (this is just for simplicity).*
Suppose that we have only one instrument, so $\text{dim}(Z) = n \times 1$. Then consider the estimator

$$\hat{\beta}_{IV} = (Z'X)^{-1}Z'Y$$

**Consistency:**

$$p \lim \hat{\beta}_{IV} = p \lim (Z'X)^{-1}Z'(X\beta + \varepsilon)$$

$$= p \lim \beta + (Z'X)^{-1}Z'\varepsilon$$

$$= \beta + [E(Z'X)]^{-1}E(Z'\varepsilon)$$

$$= \beta$$
Asymptotic normality:

$$\hat{\beta}_{IV} = \beta + (Z'X)^{-1}Z'\varepsilon$$

$$\implies \sqrt{n}(\hat{\beta}_{IV} - \beta) = \frac{1}{\sqrt{n}}\sum z_i \varepsilon_i \quad \xrightarrow{d} \quad N(0, V)$$

where $$\frac{1}{\sqrt{n}}\sum z_i \varepsilon_i \xrightarrow{d} N(0, V)$$.

The weaker the correlation between $Z$ and $X$, the larger this asymptotic variance will be. An instrument with a small $E(Z'X)$ is known as a weak instrument.
Two-stage Least Squares

- Now suppose we have \( k \) instruments (so \( \text{dim}(Z) = n \times k \)).

- Consider first regressing \( X \) on \( Z \) (that's why we called it the \textbf{first stage}) and then regressing \( Y \) on the fitted values \( \hat{X} \) from that regression. This would result in the estimator

\[
\hat{\beta}_{2SLS} = (\hat{X}'\hat{X})^{-1}\hat{X}'Y
\]

\[
= \left[(Z(Z'Z)^{-1}Z'X)'(Z(Z'Z)^{-1}Z'X)\right]^{-1}(Z(Z'Z)^{-1}Z'X)'Y
\]

- Consistency:

\[
\hat{\beta}_{2SLS} = \left[X'Z(Z'Z)^{-1}Z'Z(Z'Z)^{-1}Z'X\right]^{-1}X'Z(Z'Z)^{-1}Z'Y
\]

\[
= \left[X'Z(Z'Z)^{-1}Z'X\right]^{-1}X'Z(Z'Z)^{-1}Z'(X\beta + \varepsilon)
\]

\[
= \beta + \left[X'Z(Z'Z)^{-1}Z'X\right]^{-1}X'Z(Z'Z)^{-1}Z'\varepsilon
\]

So \( \text{plim} \hat{\beta}_{2SLS} = \beta \).
Two-stage Least Squares

- Showing asymptotic normality is the same as in the IV case, but now the asymptotic variance depends on how well the linear combination of $Z$ variables explain $X$.

- Equivalence between $\hat{\beta}_{2SLS}$ and $\hat{\beta}_{IV}$ when $Z$ is scalar:

\[
\hat{\beta}_{2SLS} = [X'Z(Z'Z)^{-1}Z'X]^{-1} X'Z(Z'Z)^{-1}Z'Y \\
= (Z'X)^{-1} [X'Z(Z'Z)^{-1}]^{-1} X'Z(Z'Z)^{-1}Z'Y \\
= (Z'X)^{-1} \tilde{Z}'Y
\]
Some Interpretation

- Remember what the IV estimator is in terms of the reduced form and the first stage:

\[ \hat{\beta}_{IV} = \frac{\hat{\pi}}{\hat{\gamma}} \]

where \( \pi \) is the coefficient on \( Z \) in the reduced form and \( \gamma \) is the coefficient on \( Z \) in the first stage. This is another way of seeing that a weak relationship between \( X \) and \( Z \) (a small \( \gamma \)) will lead to an imprecisely estimated \( \gamma \).

- If \( X \) is binary (say, our treatment, so call it \( D \)) and \( Z \) is binary (say, a randomized assignment) then the IV estimator is just

\[ \hat{\beta}_{IV} = \frac{E(Y|Z = 1) - E(Y|Z = 0)}{E(D|Z = 1) - E(D|Z = 0)} \]
Simultaneous Equations

- Model from class:
  
  \[ Y_1 + \gamma_{12} Y_2 = \alpha_1 + \beta_{11} X_1 + \beta_{12} X_2 + U_1 \]
  
  \[ \gamma_{21} Y_1 + Y_2 = \alpha_2 + \beta_{21} X_1 + \beta_{22} X_2 + U_2 \]

- Discussion of causal effects is a black hole.

- Counterfactuals are ambiguous only insofar as we fail to specify an intervention that would bring them about.

- When moving along the demand curve, does changing price cause quantity demanded to change, or does a change in quantity demanded cause price to change?
To fix ideas, rename the variables so that these two equations specify the supply curve and the demand curve respectively:

\[ Q_s + \gamma_{12} P_s = \alpha_1 + \beta_{11} X_1 + \beta_{12} X_2 + U_1 \]
\[ \gamma_{21} Q_d + P_d = \alpha_2 + \beta_{21} X_1 + \beta_{22} X_2 + U_2 \]

Because in equilibrium \( P_s = P_d \) we can plug the bottom equation into the top one and solve for \( Q = Q_s = Q_d \).

The algebra is easier in matrix notation:

\[
\Gamma \left( \begin{array}{c} Q \\ P \end{array} \right) = \left( \begin{array}{c} \alpha_1 \\ \alpha_2 \end{array} \right) + B \left( \begin{array}{c} X_1 \\ X_2 \end{array} \right) + \left( \begin{array}{c} U_1 \\ U_2 \end{array} \right)
\]
The reduced form is:

$$\begin{pmatrix} Q \\ P \end{pmatrix} = \Gamma^{-1} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} + \Gamma^{-1} B \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} + \Gamma^{-1} \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}$$

where

$$\Gamma^{-1} B = \frac{1}{1-\gamma_{12}\gamma_{21}} \begin{pmatrix} 1 & -\gamma_{12} \\ -\gamma_{21} & 1 \end{pmatrix} \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}$$

$$= \frac{1}{1-\gamma_{12}\gamma_{21}} \begin{pmatrix} \beta_{11} - \gamma_{12}\beta_{21} & \beta_{12} - \gamma_{12}\beta_{22} \\ -\gamma_{21}\beta_{11} + \beta_{21} & -\gamma_{21}\beta_{12} + \beta_{22} \end{pmatrix}$$
Exclusions

- Say we know that $X_2$ doesn’t enter the supply curve (so $\beta_{12} = 0$). That is, $X_2$ shifts the demand curve, but not the supply curve. Intuitively, this means we should be able to trace out points on the supply curve.

- We can in fact do this because if we regress $Q$ on both $X_1$ and $X_2$ the coefficient on $X_2$ will be $\frac{\beta_{12} - \gamma_{12} \beta_{22}}{1 - \gamma_{12} \gamma_{21}}$ while regressing $P$ on $X_1$ and $X_2$ will give a coefficient on $X_2$ of $\frac{-\gamma_{21} \beta_{12} + \beta_{22}}{1 - \gamma_{12} \gamma_{21}}$.

- Take the ratio of these coefficients (remember, $\beta_{12} = 0$) and you get $-\gamma_{12}$. This is precisely the slope of the supply curve.