Normal Selection Model Results from Heckman and Honoré (1990)

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The properties of the normal selection model are generated by the properties of a truncated normal model which we now establish. See Heckman and Honoré (1990). Let Z be a standard normal random variable and let $\lambda(d) \stackrel{\text{def}}{\equiv} E[Z \mid Z > d]$. For all $d \in (-\infty, \infty)$, we prove the following results:

(N-1) $\lambda(d) = \dfrac{\frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{d^2}{2}\right\}}{\Phi(-d)} > \max\{0, d\},$

(N-2) $0 < \dfrac{\partial \lambda(d)}{\partial d} = \lambda'(d) = \lambda(d)(\lambda(d) - d) < 1,$

(N-3) $\dfrac{\partial^2 \lambda(d)}{\partial d^2} > 0,$

(N-4) $0 < Var[Z \mid Z > d] = 1 + \lambda(d)d - \lambda^2(d) < 1,$

(N-5) $$\frac{\partial Var[Z \mid Z > d]}{\partial d} < 0,$$

(N-6) $$E\left[(Z - \lambda(d))^3 \mid Z > d\right] = \lambda(d)\left(2\lambda^2(d) - 3d\lambda(d) + d^2 - 1\right) = \frac{\partial^2 \lambda(d)}{\partial d^2} \quad ,$$

(N-7) $$E[Z \mid Z > d] \geq mode[Z \mid Z > d],$$

(N-8) $$\lim_{d \to -\infty} \lambda(d) = 0, \qquad \lim_{d \to \infty} \lambda(d) = \infty,$$

(N-9) $$\lim_{d \to -\infty} \frac{\partial \lambda(d)}{\partial d} = 0, \qquad \lim_{d \to \infty} \frac{\partial \lambda(d)}{\partial d} = 1,$$

(N-10) $$\lim_{d \to -\infty} Var[Z \mid Z > d] = 1, \quad \lim_{d \to \infty} Var[Z \mid Z > d] = 0.$$

Results (N-2), (N-4) and (N-5) are implications of log concavity. (N-7) is an implication of symmetry and log concavity. (N-1) and (N-3) are consequences of normality. The left hand side limits of (N-8) and (N-10) are true for any distribution. So is the right hand limit of (N-8) provided that the support of Z is not bounded on the right. The right hand limits of (N-9) and (N-10) are consequences of normality.

1 Proofs of Results (N-1) to (N-10)

The moment generating function for a truncated normal distribution with truncation point d is:

$$mgf(\beta) = e^{\beta/2} \frac{\int_{d-\beta}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}u^2\right) du}{\int_{d}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}u^2\right) du}$$

The equality in (N-1) follows from:

$$\lambda(d) = E[Z \,|Z > d] = \left.\frac{\partial mgf}{\partial \beta}\right|_{\beta=0}$$

The inequality is obvious.

By direct calculation, $\lambda'(d) = \lambda(d)(\lambda(d) - d)$. Now note that:

$$E[Z^2 \,|Z \quad > \quad d] = \frac{\partial^2 mgf}{\partial \beta^2} \,|_{\beta=0} = 1 + \lambda(d)d$$
$$E[Z^3 \,|Z \quad > \quad d] \;= \frac{\partial^3 mgf}{\partial \beta^2} \,|_{\beta=0} = \lambda(d)(2 + d^2).$$

Therefore:

$$Var[Z \,|Z > d] = 1 + \lambda^2(d) = 1 - \frac{\partial \lambda(d)}{\partial d}.$$

As Var$[Z \,|Z > d] > 0$ and $\lambda(d)(\lambda(d) - d) > 0$ by (N-1), this proves (N-2) and (N-4). To prove (N-3) notice that Var$[Z \,|Z > d] = 1 - \frac{\partial \lambda(d)}{\partial d}$, and therefore:

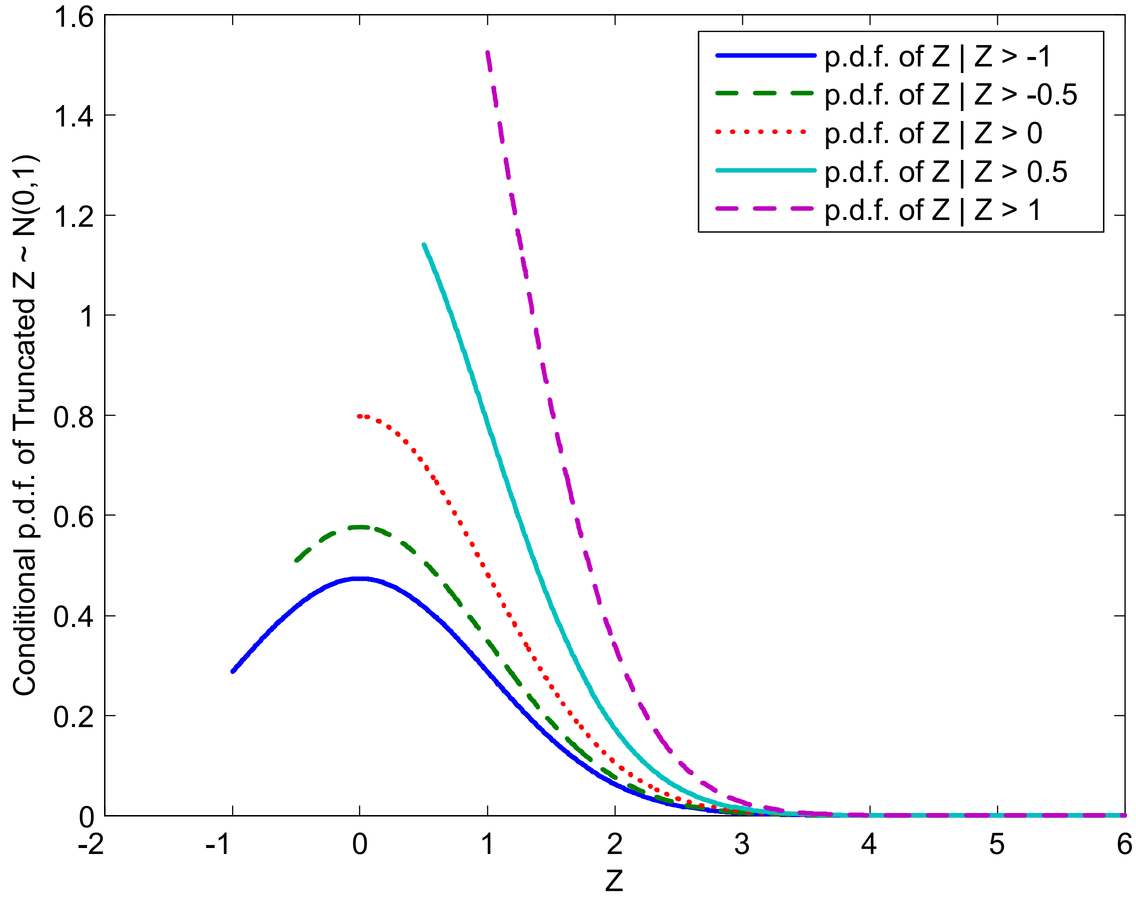
$$\frac{\partial^2 \lambda(d)}{\partial d^2} = -\frac{\partial \text{Var}[Z \,|Z > d]}{\partial d} > 0,$$

where the inequality follows from Proposition 1 in Heckman and Honore (1990). (N-5) also follows from Proposition 1, whereas (N-6) follows by direct calculation from the expression for $E[(Z - \lambda(d))^3 \,|Z > d]$. (N-7) is trivial.

(N-8) is obvious. The first part of (N-9) follows directly from ℓ' Hôpital's rule. (N-2) and (N-3) imply that $\dfrac{\partial \lambda(d)}{\partial d}$ is increasing and bounded by 1. Therefore $\lim\limits_{d \to \infty} \dfrac{\partial \lambda(d)}{\partial d}$ exists and does not exceed 1. If $\lim\limits_{d \to \infty} \dfrac{\partial \lambda(d)}{\partial d} < 1$ then $\lambda(d)$ would eventually be less than d, contradicting (N-1). This proves the second part of (N-9). (N-9) and (N-4) imply (N-10).

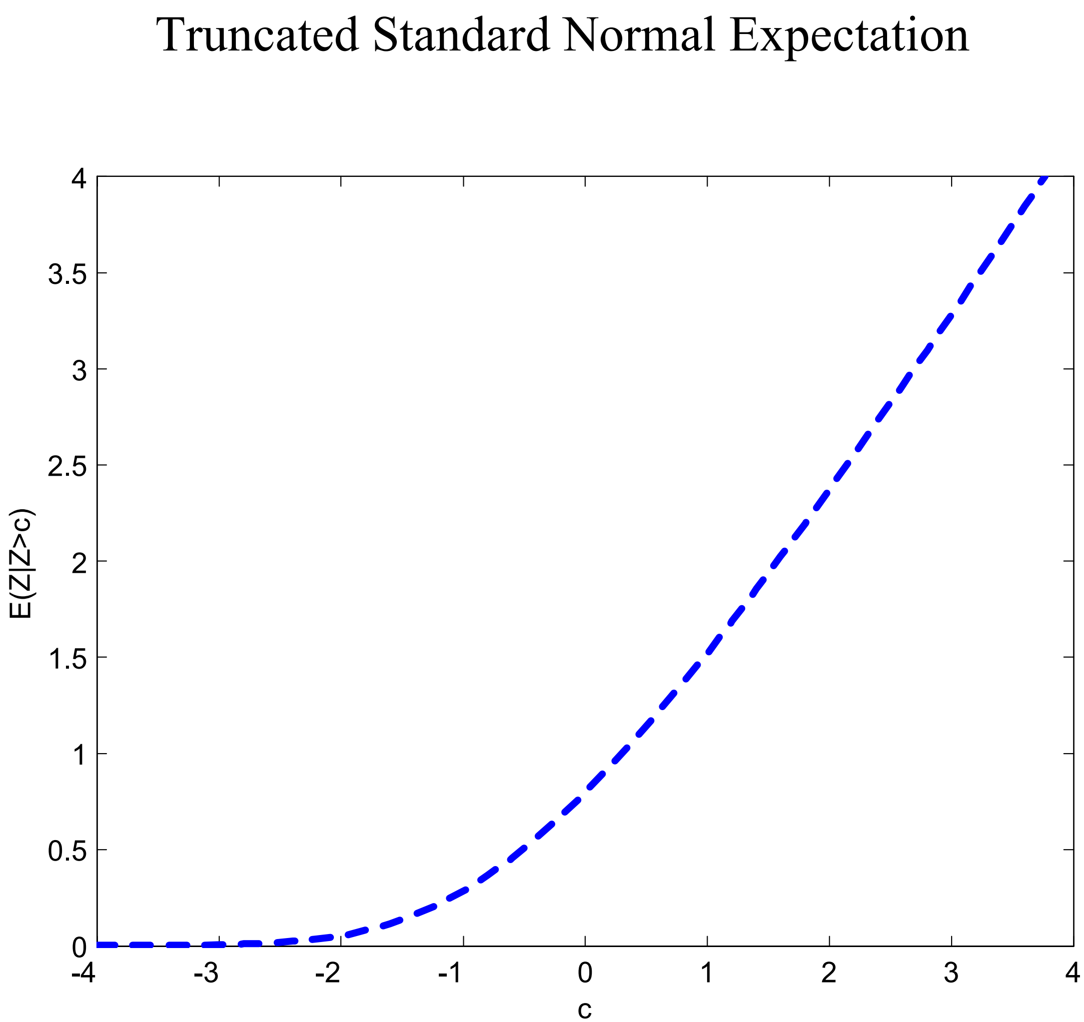
The Truncated Normal Analysis

Truncated Standard Normal Density Function



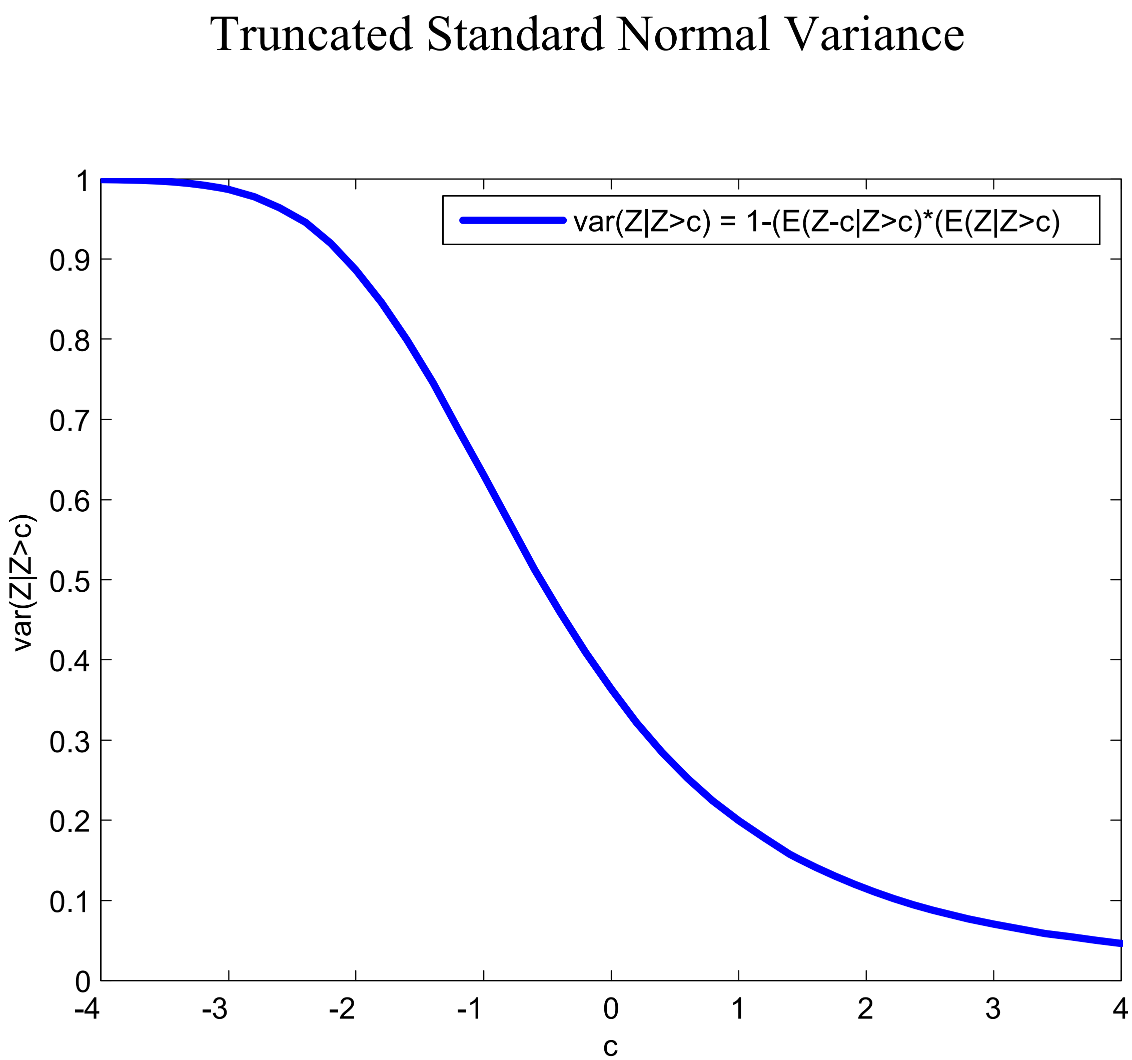
$$Z \sim N(0,1)$$

The Truncated Standard Normal Expectation



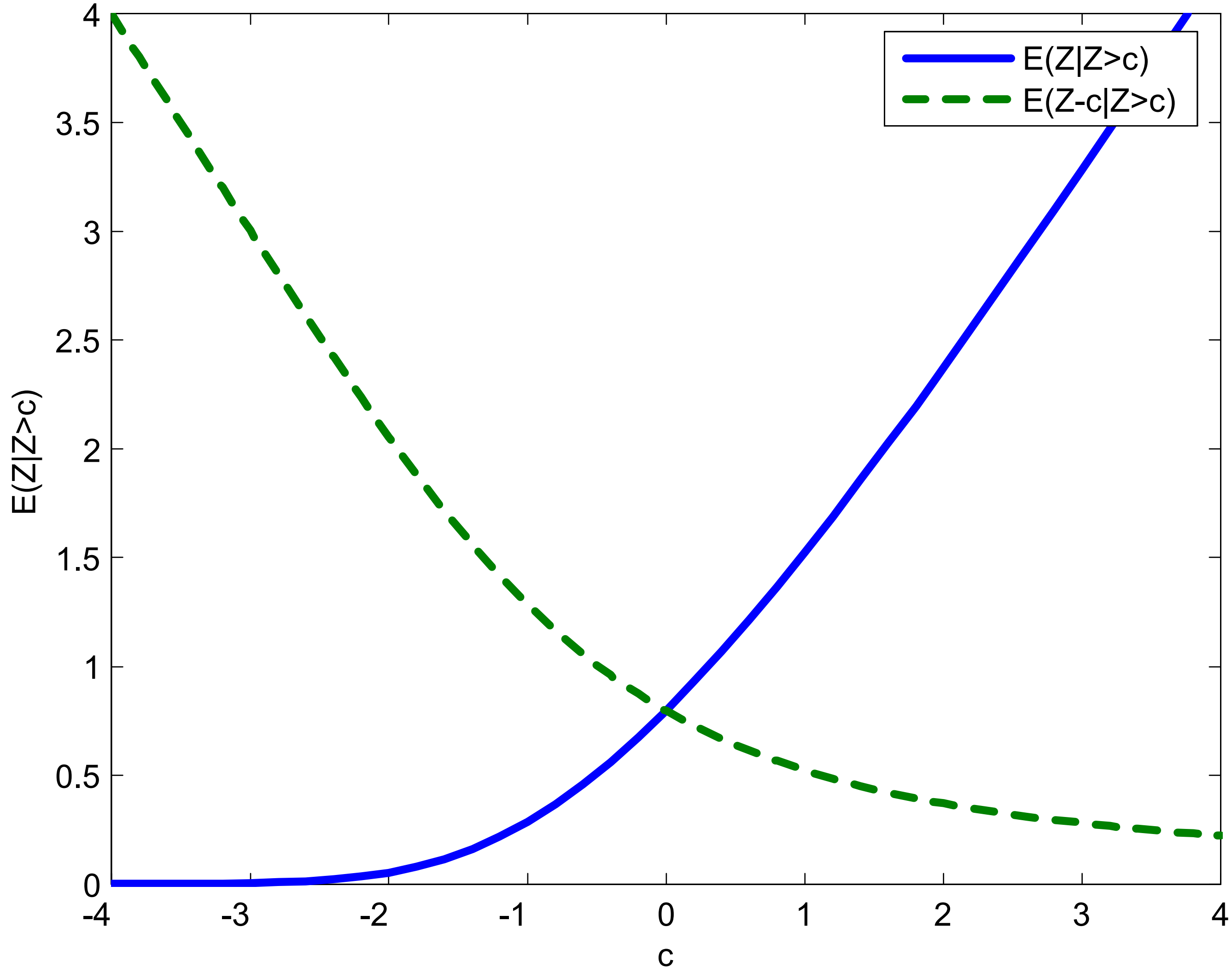
$$E[Z|Z > c]; \;\; Z \sim N(0,1)$$

The Truncated Standard Normal Variance



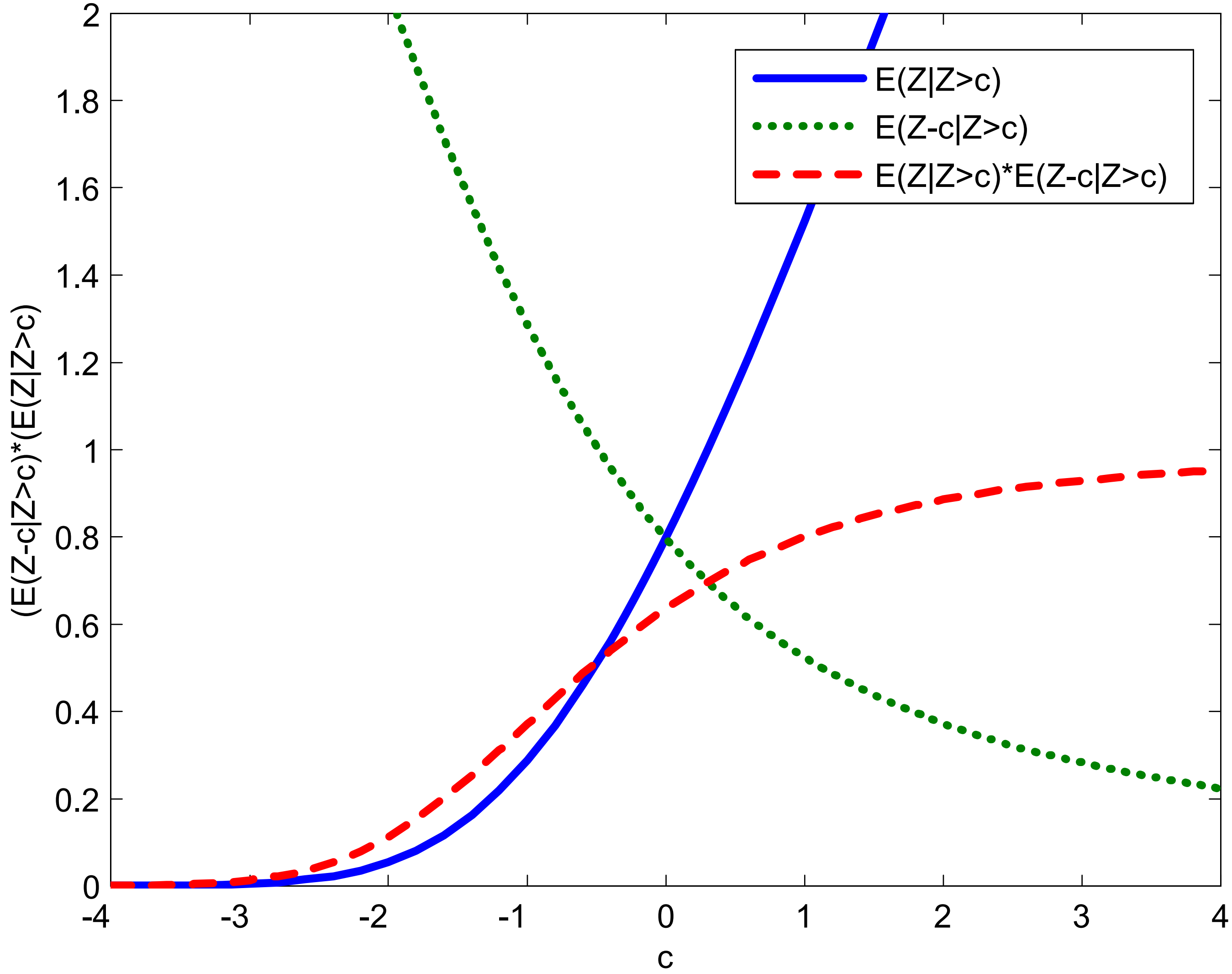
$$var(Z|Z > c); \quad Z \sim N(0,1)$$

Truncated Standard Normal Expectations



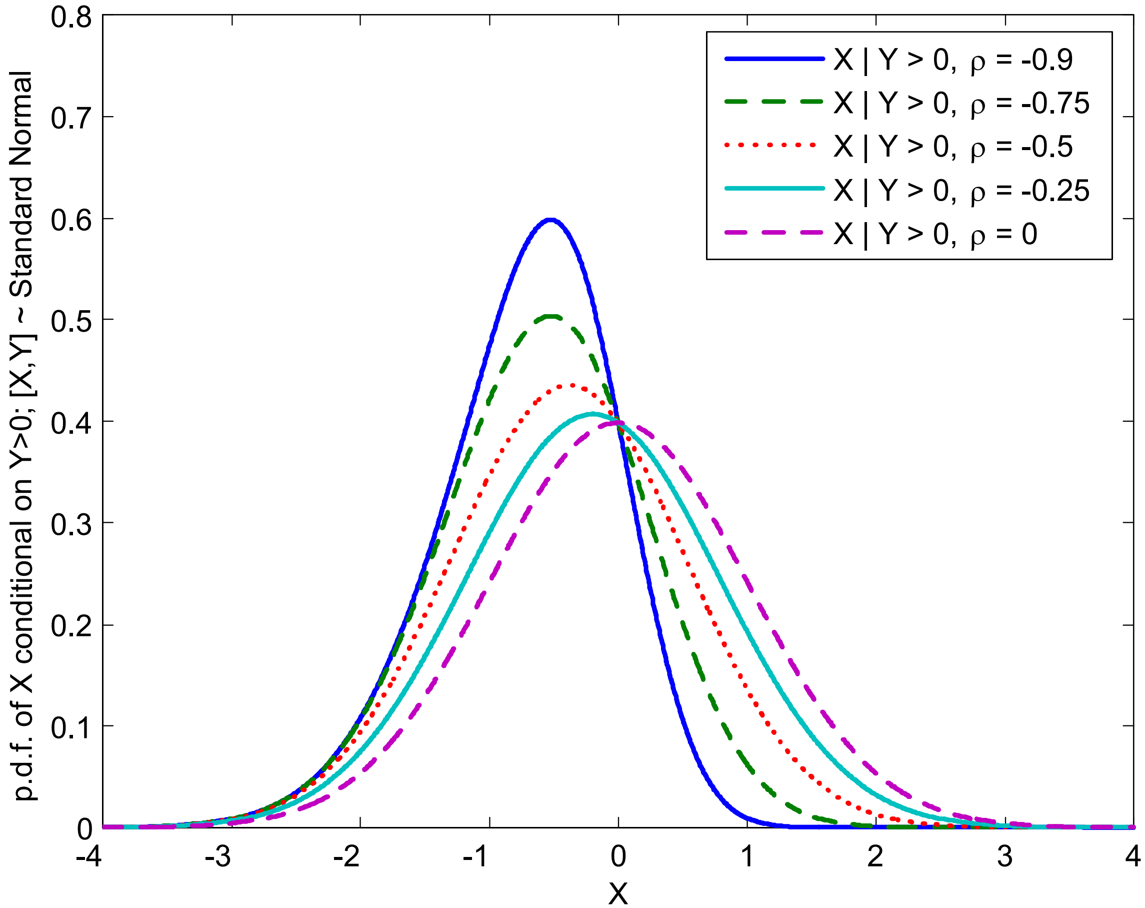
$E[Z|Z > c]$ and $E[Z - c|Z > c]$; $\quad Z \sim N(0, 1)$

Truncated Standard Normal Expectations



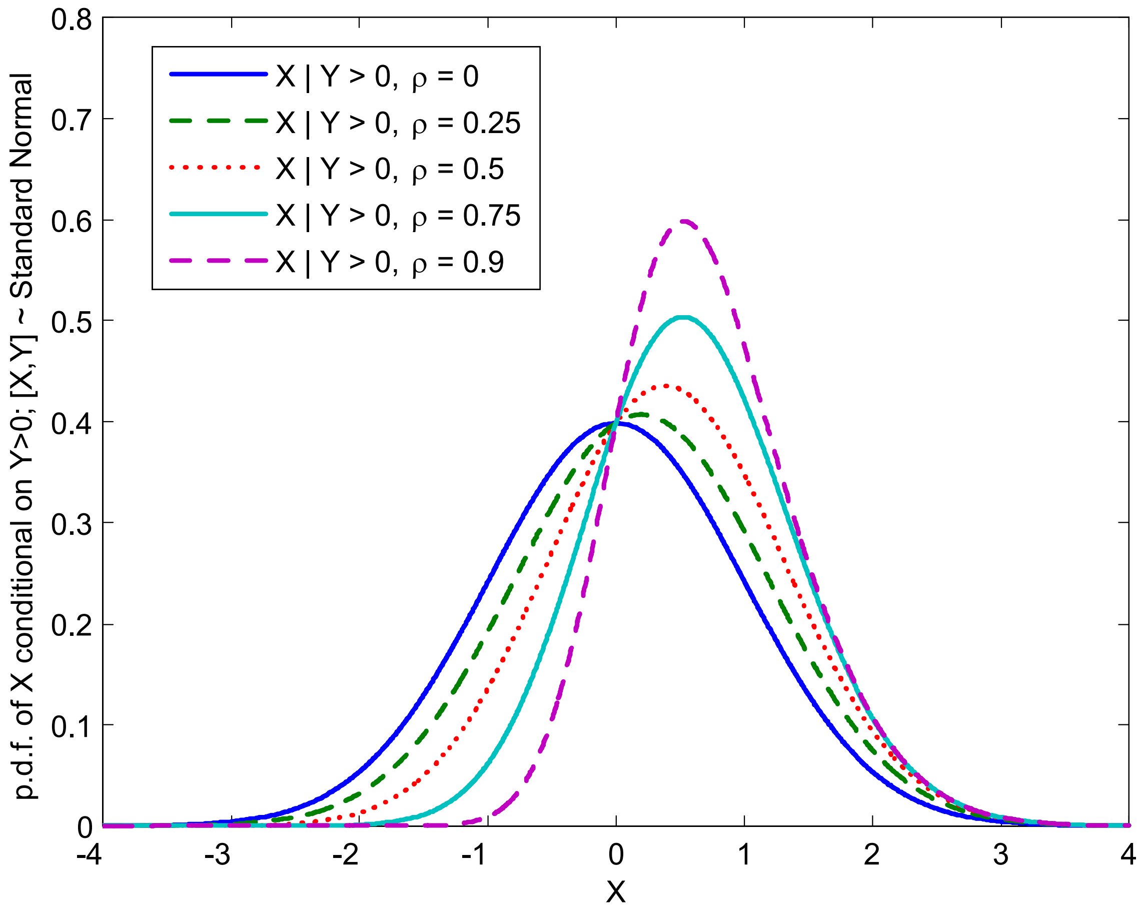
$E[Z|Z > c]$ and $E[Z - c|Z > c]$; $Z \sim N(0, 1)$

Truncated Standard Normal Density Function



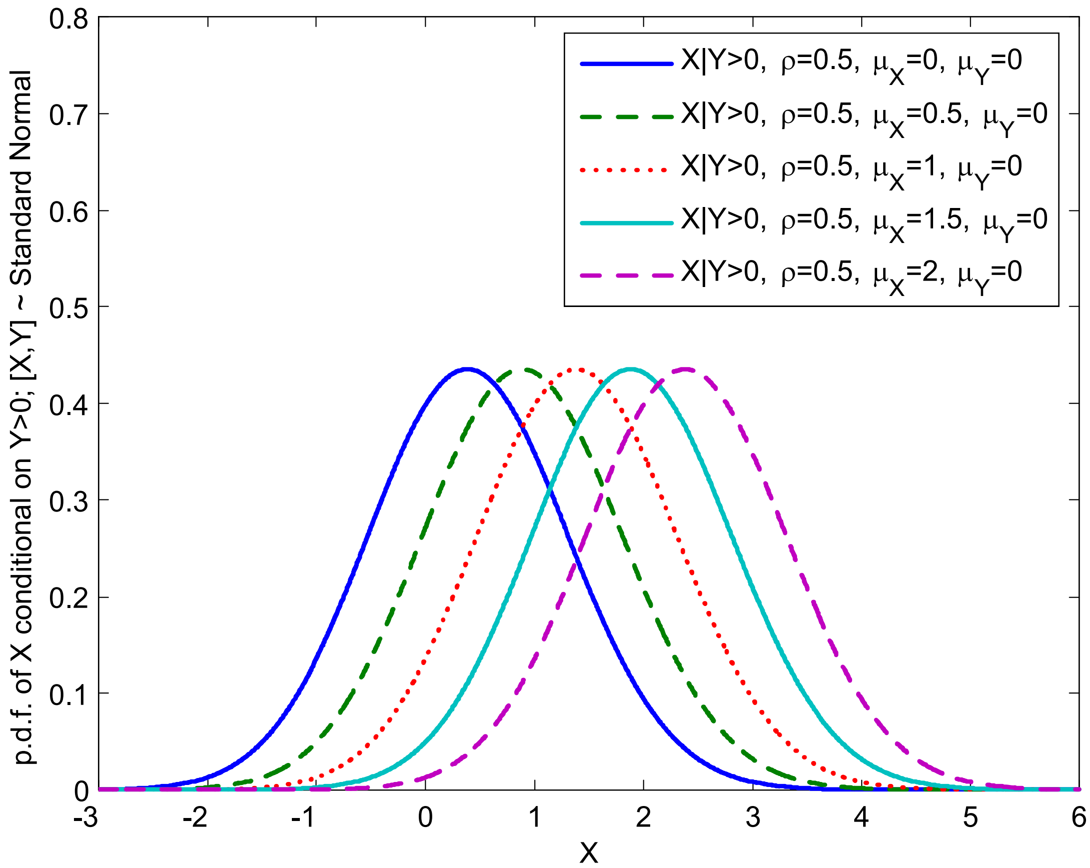
$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim N\left(\begin{matrix} 0 \\ 0 \end{matrix}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \right)$$

Truncated Standard Normal Density Function



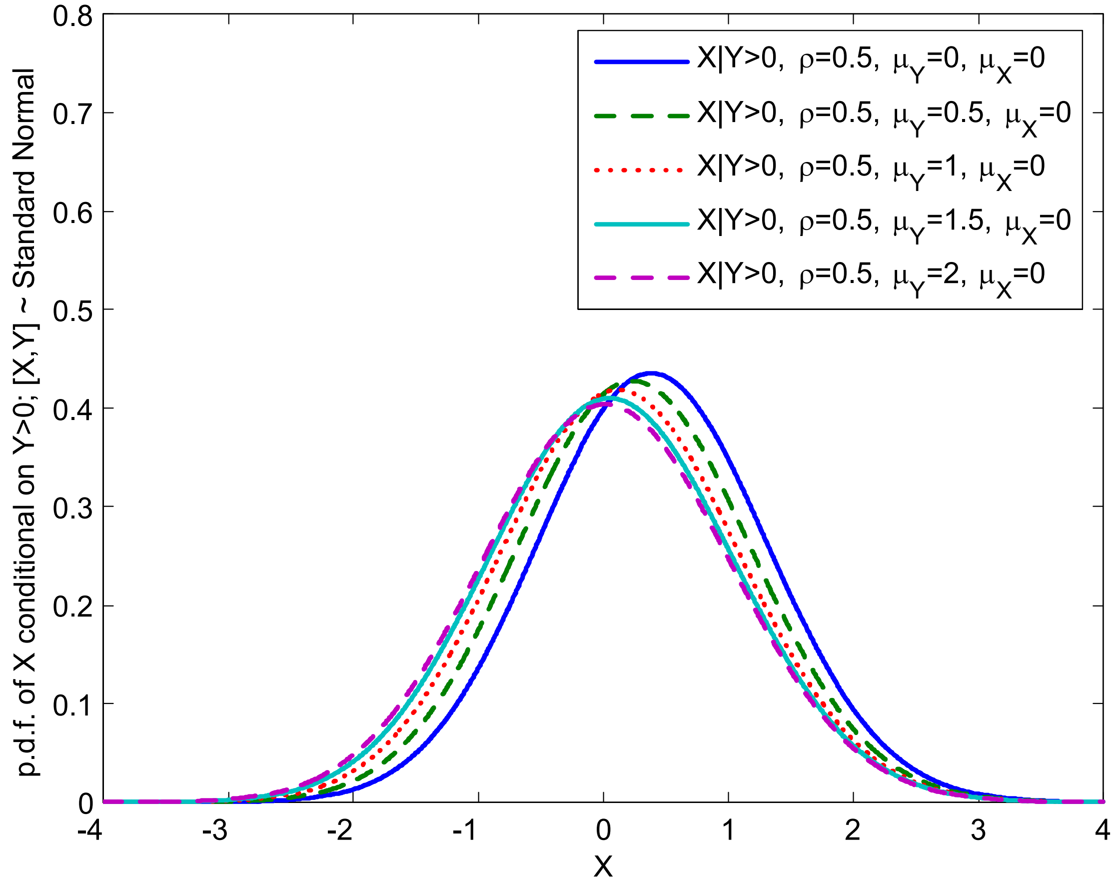
$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim N\left(\begin{matrix} 0 \\ 0 \end{matrix}, \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \right)$$

Truncated Standard Normal Density Function



$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim N\left(\begin{matrix} \mu_X \\ 0 \end{matrix}, \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \right)$$

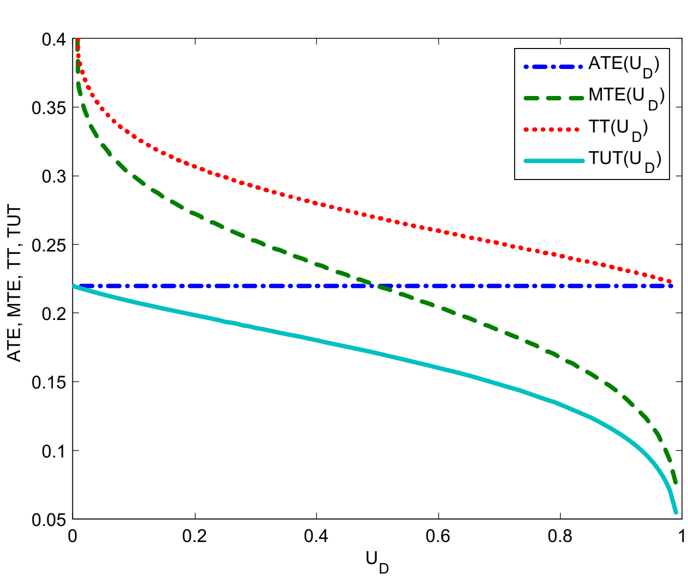
Truncated Standard Normal Density Function



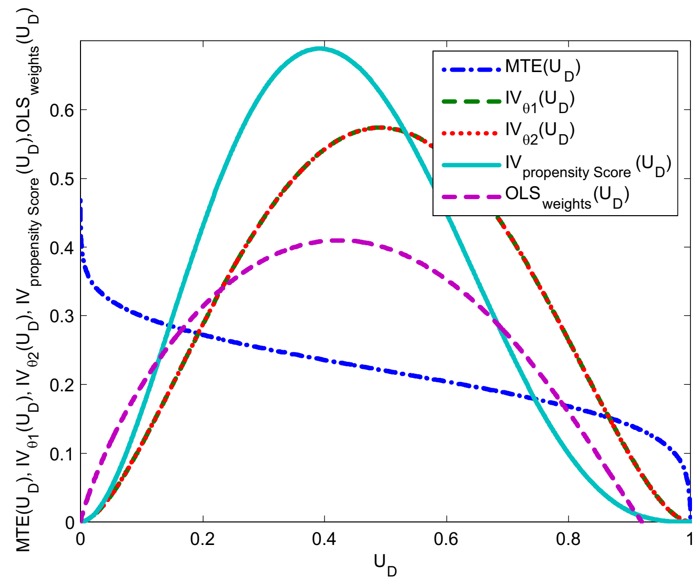
$$\begin{bmatrix} X \\ Y \end{bmatrix} \sim N\left(\begin{matrix} 0 \\ \mu_Y \end{matrix}, \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix} \right)$$

Extended Roy Model

Treatment Effects and Weights for the Extended Roy Model



Treatment Values conditional on Propensity Score



Weights conditional on Propensity Score

Model and Parameters

$$Y = D \cdot Y_1 + (1 - D) \cdot Y_0; \quad D = \mathbf{1}[\gamma Z - V > 0]; \quad V \sim N(0,1)$$

$$\gamma Z = 0.2 + 0.3 \cdot Z_1 + 0.1 \cdot Z_2; \quad U_1 = -0.012 \cdot V; \quad U_0 = 0.05 \cdot V;$$

$$Y_1 = 0.04 + 0.8 \cdot X_1 + 0.4 \cdot X_2 + U_1; \quad Y_0 = 0.22 + 0.5 \cdot X_1 + 0.1 \cdot X_2 + U_0;$$

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N\left(\begin{bmatrix} -2 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} \right); \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \sim N\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} \right)$$