Labor Supply

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One period models: \((L < 1)\)

\[
U(C, L) = \frac{C^\alpha - 1}{\alpha} + b \left( \frac{L^\varphi - 1}{\varphi} \right) \\
\alpha, \varphi < 1
\]

\(b \uparrow \implies \text{taste for leisure increases}\)
MRS at zero hours of work (Reservation Wage or Virtual Price):

\[
R = \left( \frac{\partial U}{\partial L} \right) \left( \frac{\partial U}{\partial C} \right) \bigg| L = 1, C = A
\]

\[
R = b \frac{L^{\varphi - 1}}{C^{\alpha - 1}}
\]

at \( L = 1, C = A \)

\[
R = \frac{b}{A^{\alpha - 1}}
\]

\[
\ln R = \ln b + (1 - \alpha) \ln A
\]
Set:

\[ \ln b = X \beta + \varepsilon_b \]

Assume:

\[ \varepsilon_b \sim N (0, \sigma_b^2) \]

Assume:

\[ \ln W \perp \varepsilon_b \]

\[ (X, A, W) \perp \varepsilon_b \]
Assume wage is observed for everyone. Probability that a person with assets $A$, $X$, and Wage $W$ works:

$$\Pr \left( \ln R \leq \ln W \mid X, A \right) = \Pr \left( X \beta + (1 - \alpha) \ln A + \varepsilon_b \leq \ln W \mid X, A \right) = \Pr \left( \frac{\varepsilon_b}{\sigma_b} \leq \frac{\ln W - X \beta - (1 - \alpha) \ln A}{\sigma_b} \right) = \Phi \left( C \right)$$

where

$$C \equiv \frac{\ln W - X \beta - (1 - \alpha) \ln A}{\sigma_b} \quad A > 0$$
Let

\[ D = 1 \quad \text{if person works} \]
\[ D = 0 \quad \text{otherwise} \]

\[ \Rightarrow D = 1 \left[ \ln W \geq \ln R \right] \]

\[ \Pr(\ln R \leq \ln W \mid X, A) = \Pr(D = 1 \mid X, A) \]

Take Grouped Data: Each cell has common values of \( W_i, X_i \) and \( A_i \).

\[ \hat{P}_i = \text{cell proportion working} \]

Set \( \hat{P}_i = \Phi \left( \hat{C}_i \right) \)

\[ \hat{C}_i = \frac{\ln W_i - X_i \beta - (1 - \alpha) \ln A_i}{\sigma_b} \]

inverse exists:

\[ \hat{C}_i = \Phi^{-1} \left( \hat{P}_i \right) \quad \text{(table lookup)} \]
Run Regression:

\[ \hat{C}_i \text{ on } \frac{\ln W_i - X_i \beta - (1 - \alpha) \ln A_i}{\sigma_b} \]

Coefficient on \( \ln W_i \) is \( \frac{1}{\sigma_b} \)

Coefficient on \( X \) is \( \frac{\beta}{\sigma_b} \)

Coefficient on \( \ln A \) is \( \frac{1 - \alpha}{\sigma_b} \)

Do for Logit

\[ \Pr \left( \frac{\varepsilon}{\sigma_b} \leq z \right) = \frac{e^z}{1 + e^z} \]
Linear Probability Model

\[
\Pr \left( \frac{\varepsilon}{\sigma_b} \leq z \right) = \frac{z}{z_U - z_L} \quad \text{for} \quad z_L \leq \frac{\varepsilon}{\sigma_b} \leq z_U
\]
Micro Data Analogue:
Sample size I, (Assumes we have symmetric $\varepsilon$ around zero):

$$
\mathcal{L} = \prod_{i=1}^{I} \Phi (C_i (2D_i - 1))
$$

$$
\left( \hat{\beta}, \hat{\sigma}_b, \hat{\alpha} \right) = \arg \max \ln \mathcal{L}
$$

consistent, asymptotically normal. (Likelihood is concave) Assumes we know wage for all persons, including those who work, but we don’t.

Can be nonparametric about $F_{\varepsilon_b}$ (Cosslett, Manski, Matzkin)
Digression

- $D^* = Z\gamma - V$, $D = 1(Z\gamma > V)$, assume $\text{Var}(V) = 1$.

- Can be nonparametric about $V$. Normality is not needed. Assume $Z_i, Z_j$ are continuous:

\[
\Pr (D = 1 \mid Z) = F_V (Z\gamma)
\]

\[
\frac{\partial \Pr (D = 1 \mid Z\gamma)}{\partial Z_i} \frac{\partial \Pr (D = 1 \mid Z\gamma)}{\partial Z_j}
= \frac{f_V (Z\gamma) \gamma_i}{f_V (Z\gamma) \gamma_j} = \frac{\gamma_i}{\gamma_j}
\]

- We can identify the coefficients up to scale.

- Back to text.
### Method II

Don’t know wage, but

\[
\begin{align*}
\ln W &= Z \gamma + U \\
\ln R &= X \beta + (1 - \alpha) \ln A + \varepsilon
\end{align*}
\]

\[
\begin{pmatrix} U \\ \varepsilon \end{pmatrix} \sim N \left( \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_{UU} & \sigma_{U\varepsilon} \\ \sigma_{\varepsilon U} & \sigma_{\varepsilon\varepsilon} \end{pmatrix} \right)
\]

\[
\ln R - \ln W \geq 0 \iff D = 0
\]

\[
Y_1 \equiv -X \beta - (1 - \alpha) \ln A + Z \gamma - (\varepsilon - U)
\]

\[
= \ln W - \ln R
\]
\[(X, \ln A, Z) \perp \perp (\varepsilon - U)\]

\[(\varepsilon - U) \sim N(0, \sigma_{\varepsilon\varepsilon} + \sigma_{UU} - 2\sigma_{\varepsilon U})\]

\[
\begin{align*}
\text{Var}(\varepsilon - U) &= (\sigma^*)^2 \\
\sigma^* &\equiv \sqrt{\sigma_{\varepsilon\varepsilon} + \sigma_{UU} - 2\sigma_{\varepsilon U}}
\end{align*}
\]

\[
\Pr(Y_1 \geq 0 \mid X, A, Z) = \Pr(D = 1 \mid X, \ln A, Z)
\]
One period models

Method II

For working persons

Labor Supply

Optimal Wage-Hours

Fixed Cost Models

\[
\Pr (D = 1 \mid X, \ln A, Z) = \Pr \left( \frac{-X\beta - (1 - \alpha) \ln A + Z\gamma}{\sigma^*} \geq \frac{\varepsilon - U}{\sigma^*} \right) = \Phi \left( \frac{-X\beta - (1 - \alpha) \ln A + Z\gamma}{\sigma^*} \right) = \Phi(C)
\]

\[C \equiv \frac{-X\beta - (1 - \alpha) \ln A + Z\gamma}{\sigma^*}\]

If \(Z\) and \(X\) distinct from each other and \(A\), estimate \(\frac{\gamma}{\sigma^*}, \frac{\beta}{\sigma^*}, \frac{1 - \alpha}{\sigma^*}\), can’t estimate \(\sigma^*\), \(\therefore\) get relative values.
Suppose $X$ and $Z$ have some elements in common;

$$X_c = Z_c \quad \text{elements in common}$$

$$X_d, Z_d \text{ are distinct elements in } X, Z$$

$$\frac{Y_1}{\sigma^*} = - \frac{X_d \beta_d}{\sigma^*} - \frac{X_c (\beta_c - \gamma_c)}{\sigma^*} + \frac{Z_d \gamma_d}{\sigma^*} + \frac{(1 - \alpha)}{\sigma^*} \ln A + \frac{\varepsilon - U}{\sigma^*}$$

$$\therefore \text{ identify } \frac{\beta_d}{\sigma^*}, \frac{\beta_c - \gamma_c}{\sigma^*}, \frac{\gamma_d}{\sigma^*}, \frac{1 - \alpha}{\sigma^*}$$

(The leading example of variables in common is education.)

Allows $U$ to be correlated with $\varepsilon$.

(Method II may be required anyway.)
Observe the wage only for working persons

\[
\ln W = Z\gamma + U
\]

\[
\ln R = X\beta + (1 - \alpha)\ln A + \varepsilon
\]

Assume \((X, Z, A) \perp \perp (\varepsilon, U)\)

\[
Y_1 = \ln W - \ln R = Z\gamma - X\beta - (1 - \alpha)\ln A + U - \varepsilon
\]
Letting $\tilde{\lambda}(q) = \frac{\phi(q)}{\Phi(q)}$, we have

$$E \left( \ln W \mid \ln W - \ln R \geq 0, X, Z, A \right)$$

$$= E \left( \ln W \left| \frac{Z\gamma - X\beta - (1 - \alpha) \ln A}{\sigma^*} \geq \frac{\varepsilon - \sigma U}{\sigma^*}, X, Z, A \right. \right)$$

$$= Z\gamma + \frac{\sigma_{UU} - \sigma_{U\varepsilon}}{\sigma^*} \tilde{\lambda} \left( \frac{Z\gamma - X\beta - (1 - \alpha) \ln A}{\sigma^*} \right)$$

$$C(X, A, Z) = \frac{Z\gamma - X\beta - (1 - \alpha) \ln A}{\sigma^*}$$
Remembering the Truncated Normal Random variable:

Let: \( Z \sim N(0, 1) \)

\[
E(Z|Z \geq q) = \lambda(q); \quad \lambda(q) \equiv \frac{\phi(q)}{1 - \Phi(q)} = \frac{\phi(q)}{\Phi(-q)}
\]

\[
E(Z|q \geq Z) = -E(-Z| -Z \geq -q)
\]

\[
= -\frac{\phi(-q)}{1 - \Phi(-q)} = -\frac{\phi(q)}{\Phi(q)}
\]

\[
\Rightarrow \tilde{\lambda}(q) \equiv \frac{\phi(q)}{\Phi(q)} = -E(Z|Z \leq q)
\]

and: \( E(Z|Z \geq q) = \frac{\phi(q)}{\Phi(-q)} = \lambda(q) = \tilde{\lambda}(-q) \)
Two Stage Procedures

(1) Probit on Work participation

\[
\Pr (D = 1 \mid Z, X, A) = \Pr (\ln W - \ln R \geq 0 \mid Z, X, A) = \Pr \left( \frac{Z\gamma - X\beta - (1 - \alpha) \ln A}{\sigma^*} \geq \frac{\varepsilon - U}{\sigma^*} \right | Z, X, A) = \Phi \left( \frac{Z\gamma - X\beta - (1 - \alpha) \ln A}{\sigma^*} \right)
\]

\[
\sigma^* = \left[ \text{Var} (U - \varepsilon) \right]^{1/2}
\]

\[
\therefore \text{we can estimate } C(X, A, Z)
\]

(2) Form \( \tilde{\lambda}(C) \)
Run Linear Regression
Get Consistent Estimates of

$$\gamma, \frac{\sigma_{UU} - \sigma_{U \epsilon}}{\sigma^*}$$

With one exclusion restriction (one variable in $Z$ not in $X$ or $\ln A$, say $Z_1$).
Note that using Probit if $X_d, Z_d$ are distinct elements in $X, Z$ and $X_c = Z_c$ are elements in common we can identify \( \frac{\beta_d}{\sigma^*}, \frac{\beta_c - \gamma_c}{\sigma^*}, \frac{\gamma_d}{\sigma^*}, \frac{1 - \alpha}{\sigma^*} \).

Say we recover \( \frac{\gamma_1}{\sigma^*} \) (by Probit)
Note that we have \( \gamma \) (by Wage Regression on $Z$ and $\tilde{\lambda}$)
\[ \therefore \text{know } \sigma^* \]
The estimated coefficient on $\tilde{\lambda}$ is \( \frac{\sigma_{UU} - \sigma_{U\varepsilon}}{\sigma^*} \)
\[ \therefore \text{know } \sigma_{UU} - \sigma_{U\varepsilon} \]
Look at the residuals from equations

\[ V \equiv \ln W - \left[ Z\gamma + \frac{\sigma_{uu} - \sigma_{u\varepsilon}}{\sigma^*} \tilde{\lambda}(C(X, A, Z)) \right] \]

\[ = U - \left( \frac{\sigma_{uu} - \sigma_{u\varepsilon}}{\sigma^* (\sigma_{uu})^{1/2}} \right) (\sigma_{uu})^{1/2} \tilde{\lambda}(C(X, A, Z)) \]

Let : \( \rho \equiv \frac{\sigma_{uu} - \sigma_{u\varepsilon}}{(\sigma_{uu})^{1/2} \sigma^*} \)

\[ V = U - \rho (\sigma_{uu})^{1/2} \tilde{\lambda}(C(X, A, Z)) \]

\[ = U - E(U|\ln W - \ln R \geq 0) \]

\[ \Rightarrow E(V) = 0 \]

\[ E(V^2) = \text{Var}(V) = \text{Var}(U|\ln W - \ln R \geq 0) \]
\[ E (V^2) = \sigma_{UU} \left[ (1 - \rho^2) + \rho^2 \left( 1 + \tilde{\lambda}C - \tilde{\lambda}^2 \right) \right] \]
\[ = \sigma_{UU} + \sigma_{UU}\rho^2 \left( \tilde{\lambda}C - \tilde{\lambda}^2 \right) \]

Regress

\[ \hat{V}^2 \text{ on } (\tilde{\lambda}C - C^2) \quad \text{Get } \sigma_{UU} \text{ and } \sigma_{UU}\rho^2 \]

\[ \therefore \text{ know } \rho^2 \]
Look at model:

1. Wrong variables appear in wage equation
2. Errors heteroskedastic
3. Omitted variables
Recovered Coefficients:

\[ \frac{\gamma_1}{\sigma^*} \] (Probit) \[ \gamma \] (Wage Regression) \[ \frac{\sigma_{UU} - \sigma_{U\varepsilon}}{\sigma^*} \] (Wage Regression) \[ \sigma_{UU} \] (Error\(^2\) Regression) \[ \Rightarrow \sigma^* \]
\[ \Rightarrow \sigma_{UU} - \sigma_{U\varepsilon} \]
\[ \Rightarrow \sigma_{U\varepsilon} \]
The term $\frac{\sigma_{UU} - \sigma_{U\varepsilon}}{\sigma^*}$:

\[\frac{\sigma_{UU} - \sigma_{U\varepsilon}}{\sigma^*}\]

is a Wage Regression coefficient

\[\rho \equiv \frac{\sigma_{UU} - \sigma_{U\varepsilon}}{(\sigma_{UU})^{\frac{1}{2}} \sigma^*} \left( \text{Error}^2 \text{ Regression } \right) \right\}
\Rightarrow \frac{\sigma_{UU} - \sigma_{U\varepsilon}}{\sigma^*}

⇒ 2 estimates of $\frac{\sigma_{UU} - \sigma_{U\varepsilon}}{\sigma^*}$
The term $\sigma^*$:

\[
\begin{align*}
\frac{\gamma_1}{\sigma^*} \quad \text{(Probit)} \\
\gamma \quad \text{(Wage Regression)} \\
\sigma_{UU} - \sigma_{U\varepsilon} \quad \text{(Wage Regression \& $\sigma^*$ above)} \\
\rho \equiv \frac{\sigma_{UU} - \sigma_{U\varepsilon}}{(\sigma_{UU})^{\frac{1}{2}} \sigma^*} \quad \text{(Error$^2$ Regression)} \\
\sigma_{UU} \quad \text{(Error$^2$ Regression)}
\end{align*}
\] \quad \Rightarrow \sigma^*

\Rightarrow 2 \text{ estimates of } \sigma^*$

To obtain $\sigma_{\varepsilon\varepsilon}$, we can solve

\[
(\sigma^*)^2 = \sigma_{UU} + \sigma_{\varepsilon\varepsilon} - 2\sigma_{U\varepsilon}
\]

\[
\therefore (\sigma^*)^2 + 2\hat{\sigma}_{U\varepsilon} - \hat{\sigma}_{UU} = \hat{\sigma}_{\varepsilon\varepsilon}
\]
Suppose we have no exclusion restriction, just regressors. Then we can still estimate $\gamma$, $\sigma_{UU}$, $\sigma_{\epsilon\epsilon}$ provided we substitute other information for exclusion restrictions.

$$b = \frac{\sigma_{UU} - \sigma_{U\epsilon}}{\sigma^*} = \frac{\sigma_{UU} - \sigma_{U\epsilon}}{(\sigma_{UU} + \sigma_{\epsilon\epsilon} - 2\sigma_{U\epsilon})^{\frac{1}{2}}}$$

(coefficient on $\lambda$)

$$E\left(V^2\right) = \sigma_{UU} + \sigma_{UU}\rho^2 \left(\tilde{\lambda}C - \tilde{\lambda}^2\right)$$

$$= \sigma_{UU} + \sigma_{UU} \left(\frac{\sigma_{UU} - \sigma_{U\epsilon}}{\left(\sigma_{UU}\right)^{\frac{1}{2}} \sigma^*}\right)^2 \left(\tilde{\lambda}C - \tilde{\lambda}^2\right)$$

$$= \sigma_{UU} + b^2 \left(\tilde{\lambda}C - \tilde{\lambda}^2\right)$$

$$\Rightarrow \sigma_{UU} = E\left(V^2\right) - b^2 \left(\tilde{\lambda}C - \tilde{\lambda}^2\right)$$
Normalize variables: \( \sigma_{\varepsilon \varepsilon} = 1 \) or \( \sigma_{U \varepsilon} = 0 \)

Example: \( \sigma_{U \varepsilon} = 0 \)

Then know

\[
\frac{\sigma_{UU}}{(\sigma_{UU} + \sigma_{\varepsilon \varepsilon})^{\frac{1}{2}}}
\]

\( \therefore \) can solve for \( \sigma_{\varepsilon \varepsilon} \)

Alternatively, if \( \sigma_{\varepsilon \varepsilon} = 1 \)

\[
\frac{\sigma_{UU} - \sigma_{U \varepsilon}}{(1 + \sigma_{UU} - 2\sigma_{U \varepsilon})^{\frac{1}{2}}} = \text{known}
\]

solve for \( \sigma_{U \varepsilon} \), quadratic equation – sometimes get unique root.

Note crucial role of regressor in getting full identification.
More Information:

Direct Utility Function for non workers:

\[ V_1(A_1, \varphi) \quad A_1 = \text{unearned income if person works} \]

best attainable utility for a person who doesn’t work

Indirect Utility Function:

\[ V_2(A_2, W, \varphi) \quad (W = \text{wage}) \]

best available utility given that he “works”, (which may be \( V_1 \)).

\( A_2 \) is unearned income net of money costs of work
For person who works:

If $V_2 > V_1$ person works

Index Function:

\[ Y_1 = V_2 - V_1 \quad Y_1 \geq 0 \text{ person works} \]
\[ Y_2 = H = \left( \frac{\partial V_2}{\partial W} \right) / \left( \frac{\partial V_2}{\partial A} \right) = H(A_2, W, \varphi) \]

Roy's Identity:
3 types of labor supply functions:

(a) participation
(b) $E(H|H > 0, W, A)$
(c) $E(H|W, A)$ aggregate labor supply
None estimates a labor supply function (Hicks-Slutsky). Workers free to choose hours of work. Wage $W$ is independent of hours of work. No fixed costs. Local comparison is global comparison.

Consider a simple example based on Heckman (1974),
MRS Function:

\[
\ln R = \alpha_0 + \alpha_1 A + \alpha_2 X + \eta H + \varepsilon \tag{1}
\]

\[
\ln W = Z\gamma + U
\]

\(\ln R\) defines an equilibrium value of time locus.

Labor supply \(H\) is the value that equates \(\ln W = \ln R\):

\[
\ln W = \alpha_0 + \alpha_1 A + \alpha_2 X + \eta H + \varepsilon
\]

\[
\therefore H = \frac{1}{\eta} (\ln W - \alpha_0 - \alpha_1 A - \alpha_2 X - \varepsilon)
\]

The “causal effect” of \(\ln\) (wage) on labor supply is \(\frac{1}{\eta}\) (holding \(A, X\) and \(\varepsilon\) constant).

This is a Hicks-Slutsky effect.
E.g.

\[
\frac{\partial H}{\partial \ln W} = S + H \frac{\partial H}{\partial Y},
\]

\[
\frac{1}{\eta} = \frac{\partial H}{\partial \ln W} = WS + (WH) \frac{\partial H}{\partial Y}.
\]

If \(\eta\) is constant, then as \(H \uparrow\), for a fixed \(W\), \(S \uparrow\) (more substitution).

As \(W \uparrow\), \(S + H \frac{\partial H}{\partial Y} \downarrow\), so the Hicks-Slutsky effect declines (net labor supply becomes more inelastic in this sense).
Figure: Value of Time
Define

\[ Y_1 = \ln W - \ln R = Z\gamma + U - \alpha_0 - \alpha_1 A - \alpha_2 X - \varepsilon \]
\[ = (Z\gamma - \alpha_0 - \alpha_1 A - \alpha_2 X) + (U - \varepsilon). \]

Hours of work then are:

\[ Y_2 = H = \frac{1}{\eta} Y_1 \quad \text{if } Y_1 \geq 0 \]
\[ Y_3 = \ln W = Z\gamma + U \]

\[ \text{Var}(U - \varepsilon) = (\sigma^*)^2 \]
Population Labor Supply is generated from $Y_2$

\[
E \left( \frac{H}{Y_1 > 0, Z, A, X} \right) = \frac{1}{\eta} E \left( \ln W - \ln R \mid \ln W - \ln R > 0, Z, A, X \right) \]

\[
= Z\gamma - \alpha_0 - \alpha_1 A - \alpha_2 X
\]

\[
= \frac{1}{\eta} E \left( \frac{U - \varepsilon}{\sigma^*} \mid \frac{U - \varepsilon}{\sigma^*} > -(Z\gamma - \alpha_0 - \alpha_1 A - \alpha_2 X) \right) \]

\[
= Z\gamma - \alpha_0 - \alpha_1 A - \alpha_2 X
\]
Let

\[ C \equiv \frac{Z\gamma - \alpha_0 - \alpha_1 A - \alpha_2 X}{\sigma^*} \]

\[
\frac{1}{\eta} E(H|Y_1 > 0, Z, A, X) \\
= \frac{1}{\eta} E(Y_1|Y_1 > 0, Z, A, X) \\
= \frac{(Z\gamma - \alpha_0 - \alpha_1 A - \alpha_2 X)}{\eta} \\
+ \frac{1}{\eta} E(U - \varepsilon|U - \varepsilon > -(Z\gamma - \alpha_0 - \alpha_1 A - \alpha_2 X), Z, A, X) 
\]
\[
\begin{align*}
&= \frac{C}{\eta} \sigma^* + \frac{\sigma^*}{\eta} E \left( \left| \frac{U - \varepsilon}{\sigma^*} \right| \frac{U - \varepsilon}{\sigma^*} > -C, Z, A, X \right) \\
&= \frac{C}{\eta} \sigma^* + \frac{1}{\eta} \text{Cov}(U - \varepsilon, U - \varepsilon) \tilde{\lambda}(C) \\
&= \frac{C}{\eta} \sigma^* + \frac{\sigma^*}{\eta} \tilde{\lambda}(C) \\
&= \frac{\sigma^*}{\eta} \left( C + \tilde{\lambda}(C) \right)
\end{align*}
\]

\[
E(\ln W \mid Y_1 > 0, Z) = E(\ln W \mid \ln W - \ln R > 0, Z) = Z \gamma + \sigma^* E\left( \frac{U}{\sigma^*} \left| \frac{U - \varepsilon}{\sigma^*} > -C, Z \right. \right) \\
= Z \gamma + \sigma^* \text{Cov}(U, U - \varepsilon) \tilde{\lambda}(C)
\]
Assume Regressors are available:
\[ \therefore \text{We can estimate } \gamma \text{ from linear regression } \ln W \text{ on } Z\gamma \text{ and } \tilde{\lambda}(C) \text{ using the known steps:} \]

(1) From participation equation, we can use probit to estimate

\[
\Pr (D = 1 | Z, A, X) = \Pr (Y_1 > 0 | Z, A, X) = \Phi \left( \frac{Z\gamma - \alpha_0 - \alpha_1 A - \alpha_2 X}{\sigma^*} \right) = \Phi (C)
\]

\[ \therefore \text{We know } \frac{\gamma}{\sigma^*}, \frac{\alpha_0}{\sigma^*}, \frac{\alpha_1}{\sigma^*}, \frac{\alpha_2}{\sigma^*} \text{ if } X \neq A \neq Z \text{ or set of common and distinct coefficients depending on } X, A, Z \text{ elements.} \]

\[ \therefore \text{We know } C. \]

(2) Form \( \tilde{\lambda}(C) \).

(3) From the Wage Regression of \( \ln W \) on \( Z \) and \( \tilde{\lambda}(C) \).
:: we know $\gamma, \frac{\text{Cov}(U, U-\varepsilon)}{\sigma^*}$.

Thus we know

$$\frac{\text{Cov}(U, U - \varepsilon)}{\sigma^*} = \frac{\sigma_{UU} - \sigma_{U\varepsilon}}{(\sigma_{UU} + \sigma_{\varepsilon\varepsilon} - 2\sigma_{U\varepsilon})^{1/2}}.$$  

(4) From Error Regression: $\hat{V}^2$ on constant and $(\tilde{\lambda}C - C^2)$, we estimate:

$$E(V^2) = \sigma_{UU} + \sigma_{UU} \rho^2(\tilde{\lambda}C - \tilde{\lambda}^2)$$

:: know $\sigma_{UU}, \rho^2$

Same position as before. Further identification of parameters is possible due to hours of work:

(5) From hours of work data we have a proportionality restriction
\[ E(H|Y_1 > 0, Z, A, X) = \left( \frac{\sigma^*}{\eta} \right) C + \frac{\sigma^*}{\eta} \tilde{\lambda}(C) \]

but from employment (participation) equation we know \( C \)

\[ \therefore \text{can estimate } \frac{\sigma^*}{\eta} \]
(6) Using \( \frac{\text{Cov}(U, U-\varepsilon)}{\sigma^*} \) from Wage Regression and covariance assumptions, we obtain:

\[
\text{if } \sigma_{U\varepsilon} = 0 \Rightarrow \frac{\text{Cov}(U, U-\varepsilon)}{\sigma^*} = \frac{\sigma_{UU}}{\left(\sigma_{UU} + \sigma_{\varepsilon\varepsilon}\right)^{1/2}}
\]

but \( \sigma_{UU} \) was obtained by Error Regression

\[ \therefore \text{ know } \sigma^* \text{ and } \sigma_{\varepsilon\varepsilon} \]

By The hours of Work Regression (5) we obtain \( \frac{\sigma^*}{\eta} \)

\[ \therefore \text{ know } \eta \]

Similarly if \( \sigma_{\varepsilon\varepsilon} = 1 \Rightarrow \sigma_{U\varepsilon} \) known

\( \sigma_{U\varepsilon} \) known (sometimes; multiple roots)

\[ \therefore \text{ we have that all parameters are identified.} \]
(7) If there is one variable in $Z$ not in $(X, A)$, say $Z_1$, from coefficient on $Z_1$ in $E(H|Y_1 > 0, Z, A, X)$, we obtain:

$$E(H|Y_1 > 0, Z, A, X)$$

\[
= \left( \frac{\sigma^*}{\eta} \right) C + \frac{\sigma^*}{\eta} \tilde{\lambda}(C) \\
= \left( \frac{\sigma^*}{\eta} \right) \left( C + \tilde{\lambda}(C) \right) \\
= \left( \frac{\sigma^*}{\eta} \right) \left( \frac{Z \gamma - \alpha_0 - \alpha_1 A - \alpha_2 X}{\sigma^*} + \tilde{\lambda}(C) \right) \\
= Z \frac{\gamma}{\eta} + \frac{-\alpha_0 - \alpha_1 A - \alpha_2 X}{\eta} + \left( \frac{\sigma^*}{\eta} \right) \tilde{\lambda}(C)
\]
but from Wage Regression (3), we obtain $\gamma$

∴ can estimate $\eta$

but the coefficient of $\tilde{\lambda}(C)$ is $\left(\frac{\sigma^*}{\eta}\right)$

∴ can estimate $\sigma^*$.

Alternatively, we can determine $\eta$ if

$$\text{Cov}(U, \varepsilon) = 0 \quad \text{or} \quad \text{Var}(\varepsilon) = 1.$$
Selection Bias in Labor Supply
Assume $\gamma_j > 0$

$$\frac{\partial E(H \mid Y_1 > 0, Z, A, X)}{\partial Z_j} = \frac{\partial \left[ \left( \frac{\sigma^*}{\eta} \right) \left( C + \tilde{\lambda}(C) \right) \right]}{\partial Z_j}$$

$$= \frac{\partial \left[ Z \gamma \eta + \frac{-\alpha_0 - \alpha_1 A - \alpha_2 X}{\eta} + \left( \frac{\sigma^*}{\eta} \right) \tilde{\lambda} \left( \frac{Z \gamma \eta + \frac{-\alpha_0 - \alpha_1 A - \alpha_2 X}{\eta}}{\sigma^*} \right) \right]}{\partial Z_j}$$
But

\[
\frac{\partial \tilde{\lambda}(q)}{\partial q} = \frac{\partial \frac{\phi(q)}{\Phi(q)}}{\partial q} = -q \frac{\phi(q)}{\Phi(q)} - \frac{\phi^2(q)}{\Phi^2(q)} = -\tilde{\lambda}(q)\left(q + \tilde{\lambda}(q)\right)
\]

\[
= \frac{\gamma_j}{\eta} - \frac{1}{\eta} \tilde{\lambda}(\tilde{\lambda} + C) \gamma_j
\]

\[
= \left(\frac{\gamma_j}{\eta}\right) (1 - \tilde{\lambda}(\tilde{\lambda} + C))
\]

\[
0 < 1 - \tilde{\lambda}(\tilde{\lambda} + C) < 1 \therefore < \frac{\gamma_j}{\eta} \text{ downward bias.}
\]

\[
\frac{\partial H}{\partial Z_j} = \frac{\partial E(H \mid Y_1 > 0, Z, A, X)}{\partial \ln W} \leq \frac{1}{\eta}
\]

\[
\therefore \text{downward biased}
\]
\[ E(H|Y_1 > 0, Z, A, X) \]
\[
= \left( \frac{\sigma^*}{\eta} \right) C + \frac{\sigma^*}{\eta} \tilde{\lambda}(C) \\
= \left( \frac{\sigma^*}{\eta} \right) \left( C + \tilde{\lambda}(C) \right) \\
= \left( \frac{\sigma^*}{\eta} \right) \left( Z\gamma - \alpha_0 - \alpha_1 A - \alpha_2 X \right) + \tilde{\lambda}(C) \\
= Z\gamma \frac{\eta}{\eta} + \frac{-\alpha_0 - \alpha_1 A - \alpha_2 X}{\eta} + \left( \frac{\sigma^*}{\eta} \right) \tilde{\lambda}(C) \]
but from Wage Regression (3), we obtain $\gamma$

$\therefore$ can estimate $\eta$

but the coefficient of $\tilde{\lambda}(C)$ is $\left(\frac{\sigma^*}{\eta}\right)$

$\therefore$ can estimate $\sigma^*$. 
Aggregate Labor Supply

\[ \text{ALS} \equiv \Phi(C) \cdot E(H|Y_1 > 0) + \Phi(-C) \cdot E(H|Y_1 < 0) \]

\[ = \Phi(C) \left[ \left( \frac{\sigma^*}{\eta} \right) (C + \tilde{\lambda}(C)) \right] + \Phi(-C) \cdot [0] \]

\[ = \left( \frac{\sigma^*}{\eta} \right) \left[ \Phi(C)C + \frac{1}{\sqrt{2\pi}} e^{-C^2/2} \right] \]
\[
\frac{\partial E(H_{ALS} \mid Z, A, X)}{\partial C} = \frac{\sigma^*}{\eta} \left[ \Phi(C) + \frac{Ce^{-\frac{1}{2}C^2/2}}{\sqrt{2\pi}} - \frac{Ce^{-C^2/2}}{\sqrt{2\pi}} \right]
\]

\[
\frac{\partial E(H_{ALS} \mid Z, A, X)}{\partial Z_j} = \frac{\partial E(H \mid Z, A, X)}{\partial C} \frac{\partial C}{\partial Z_j} = \frac{\gamma_j}{\eta} \Phi(C)
\]
Obviously aggregate labor supply more elastic because of entry or exit:

$$\frac{1}{E(H|Y_1 > 0, W)P( Y_1 > 0|W)} \cdot \frac{\partial \{E(H|Y_1 > 0, W)P( Y_1 > 0|W)\}}{\partial \ln W} \geq \left(\frac{\partial E(H|Y_1 > 0, W)}{\partial \ln W}\right) \frac{1}{E(H|Y_1 > 0, W)}, \text{ and}$$

$$\frac{\partial \ln E(H|Y_1 > 0, W)}{\partial \ln W} + \frac{\partial \ln P(Y_1 > 0|W)}{\partial \ln W}$$

> \frac{\partial \ln E(H|Y_1 > 0, W)}{\partial \ln W}$$

Many ways to estimate model.
Labor Supply with Optimal Wage-Hours Contracts (Lewis, 1969; Rosen, 1974; Tinbergen, 1951, 1956)

**Figure:** Optimal Wage
If $Y(h)$ is earnings, and $Y'(h)$ is marginal wage, 
Virtual Income $= Y(h) - Y'(h) h + A$,
where $h = h(Y(h), Y(h) - Y'(h) h + A, \nu)$
Any equilibrium calculation use slope at zero hours of work.

$$\ln M(0, A) \leq \ln W(0) \quad \text{doesn’t work}$$

Equilibrium:

$$\ln M(h, A) = \ln W(h) \quad \text{person works}$$

Can use estimated $\ln M(0, A)$ to price out goods that previously were not purchased.
Fixed Cost Models: Fixed Money Cost (Cogan; 1981 Econometrica)

Figure: Fixed Cost Models
Introduce fixed money cost: given the wage, the worker selects a minimum number of hours.

1. Solve for $W_d$ and $H_d$ that causes the worker to be indifferent between work and no work

$$V(A - F, W_d, \varphi) = U(A, 1, \varphi)$$

Solve for $W_d$

If no solution, person doesn’t work

2. Minimum number of hours $H_d = V_W / V_A$

$$H_d = H_d(A - F, W_d, \varphi)$$

$$W_d = W_d(A - F, \varphi)$$

$$H = H(A - F, W, \varphi)$$
Index Function Model.

\[ Y_1 = H - H_d \]
\[ Y_2 = H \]

Observe \( Y_2 \) only when \( Y_1 > 0 \)

Example: (assume wage is known).

\[ H = X\beta + W\eta + \varepsilon_1 \quad \text{functional form assumptions} \]
\[ H_d = X\tau + \varepsilon_2 \]

Pr (consumer works) =

\[ \Pr(H - H_d > 0|X, W) \]
\[ = \Pr(X\beta + W\eta + \varepsilon_1 - X\tau - \varepsilon_2 > 0) \]
\[ = \Pr(X(\beta - \tau) + W\eta > \varepsilon_2 - \varepsilon_1) \]
\[ \sigma^* \equiv \sqrt{\text{Var}(\varepsilon_2 - \varepsilon_1)} \]
Assume normality and we can identify

\[ \frac{\beta - \tau}{\sigma^*} \text{and} \frac{\eta}{\sigma^*} \]

From hours of work equation we know \( \eta \)

\( \therefore \) know \( \sigma^* \)

\[
E(H|H - H_d > 0, X, W) = X\beta + W\eta + E(\varepsilon_1|X(\beta - \tau) + W\eta > \varepsilon_2 - \varepsilon_1, X, W)
\]

\[
= X\beta + W\eta + \frac{\text{Cov}(\varepsilon_1, \varepsilon_2)}{\sigma^*} \tilde{\lambda}(C)
\]

\[
C = \left(\frac{X(\beta, \tau) + W\eta}{\sigma^*}\right)
\]

Know \( \eta, \beta \). \( \therefore \) know \( \tau \)
Know $\text{Var}(\varepsilon_1)$, know $\sigma^*$

$\therefore$ know $\text{Cov}(\varepsilon_1, \varepsilon_2)$

$\therefore$ know $\text{Var}(\varepsilon_2) = (\sigma^*)^2 - \text{Var}(\varepsilon_1) + 2 \text{Cov}(\varepsilon_1, \varepsilon_2)$
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Cogan doesn’t measure fixed costs
Figure: Fixed vs. Not Fixed
Null: Departure from simple proportionality model Cogan’s test conditional on function form.
Broken line Budget Constraint (2 part prices; negative income tax data)

Two cases:

A. Know which interval person is in (no measurement error for hours)

B. Don’t know which branch (income tax data)
Figure: Which Branch?
Take case 1:

1. \( M(A, 1, \varepsilon) \geq W_1 \) does a person work?

2. Person is interior in interval \((0, \bar{h})\) if

\[
M(A + W_1 \bar{h}, 1 - \bar{h}, \varepsilon) \geq W_1 \\
M(A, 1, \varepsilon) < W_1
\]

3. In equilibrium at \( \bar{h} \) if \( W_2 \leq M(A + W_1 \bar{h}, 1 - \bar{h}, \varepsilon) \leq W_1 \)

4. Works beyond \( \bar{h} \) if \( M(A + W_1 \bar{h}, 1 - \bar{h}, \varepsilon) \leq W_2 \)
Example:

\[ U = \frac{C^\alpha - 1}{\alpha} + b \left( \frac{L^{\varphi} - 1}{\varphi} \right) \quad \alpha < 1, \varphi < 1 \]

\[ b \frac{L^{\varphi-1}}{C^{\alpha-1}} = MRS \]

at zero hours work \((L = 1)\)
\[
\ln b = X\beta + \varepsilon \\
\ln b + (\phi - 1) \ln(1) - (\alpha - 1) \ln A \geq \ln W_1 \text{ (doesn’t work)}
\]

\[
X\beta + \varepsilon - (\alpha - 1) \ln A \geq \ln W_1
\]

\[
\varepsilon \geq \ln W_1 + (\alpha - 1) \ln A - X\beta
\]

\[
E(\varepsilon^2) = \sigma_\varepsilon^2
\]

\[
\frac{\varepsilon}{\sigma_\varepsilon} \geq \frac{\ln W_1 + (\alpha - 1) \ln A - X\beta}{\sigma_\varepsilon}
\]

condition for not working

estimate: \( \sigma_\varepsilon, \alpha, \beta \)
(2) Interior in the first branch \((0, \bar{h})\)

\[
\frac{X \beta + (\varphi - 1) \ln(1 - \bar{h}) - (\alpha - 1) \ln(A + W_1 \bar{h}) - \ln W_1}{\sigma_\varepsilon} \geq \frac{-\varepsilon}{\sigma_\varepsilon}
\]

\[
\frac{X \beta + (\varphi - 1) \ln(1) - (\alpha - 1) \ln(A) - \ln W_1}{\sigma_\varepsilon} \leq \frac{-\varepsilon}{\sigma_\varepsilon}
\]

Use principle of index function, with variation in \(\bar{h}\), we identify \(\varphi\) and \(\beta\).
(3) Kink Equilibrium:

\[
\ln W_2 \leq X\beta + \varepsilon + (\varphi - 1) \ln(1 - \bar{h}) - (\alpha - 1) \ln(A + W_1 \bar{h}) \leq \ln W_1
\]

\[
\frac{\ln W_2 - X\beta - (\varphi - 1) \ln(1 - \bar{h}) + (\alpha - 1) \ln(A + W_1 \bar{h})}{\sigma_\varepsilon} \leq \frac{\varepsilon}{\sigma_\varepsilon}
\]

\[
\leq \frac{\ln W_1 - X\beta - (\varphi - 1) \ln(1 - \bar{h}) + (\alpha - 1) \ln(A + W_1 \bar{h})}{\sigma_\varepsilon}
\]
\[
\Pr(h = \bar{h} \mid W_1, W_2, X, A) =
\]
\[
\Phi \left( \frac{\ln W_1 - X \beta - (\varphi - 1) \ln(1 - \bar{h}) - (\alpha - 1) \ln(A + W_1 \bar{h})}{\sigma_\varepsilon} \right) - \Phi \left( \frac{\ln W_2 - X \beta - (\varphi - 1) \ln(1 - \bar{h}) + (\alpha - 1) \ln(A + W_1 \bar{h})}{\sigma_\varepsilon} \right)
\]
(4) In second branch interior

\[
\Pr \left( \frac{X\beta + (\eta - 1) \ln(1 - \bar{h}) - (\alpha - 1) \ln(A + W_1 \bar{h}) - \ln W_2}{\sigma_\varepsilon} \leq \frac{\varepsilon}{\sigma_\varepsilon} \right) = \Phi \left( \frac{\ln W_2 - X\beta - (\eta - 1) \ln(1 - \bar{h}) + (\alpha - 1) \ln(A + W_1 \bar{h})}{\sigma_\varepsilon} \right)
\]

\[\therefore \text{solve out hours of work.}\]
Hours of work (standard case)

\[ U = \frac{C^\alpha - 1}{\alpha} + b \left( \frac{L^\varphi - 1}{\varphi} \right) \]

\[ A = \text{nonmarket income} \]
\[ W = \text{wage} \]
\[ C = W(1 - L) + A \]
Set $\varphi = \alpha$,

Set

$$h = \frac{\left(\frac{b}{W}\right)^{\frac{1}{\alpha-1}} - A}{\left(\frac{b}{W}\right)^{\frac{1}{\alpha-1}} + W}$$

estimating equation:

$$\ln \left(\frac{Wh + A}{1 - h}\right) = \frac{\ln W}{1 - \alpha} - \frac{X\beta}{1 - \alpha} - \frac{\varepsilon}{1 - \alpha}$$
Interior in Branch 1:

\[
E \left[ \ln \left( \frac{W_1 h + A}{1 - h} \right) \middle| W_1, A \right] = \frac{\ln W_1}{1 - \alpha} - \frac{X \beta}{1 - \alpha}
\]

\[
- \frac{1}{1 - \alpha} E \left[ \frac{X \beta + (\eta - 1) \ln (1 - \bar{h})}{-(\alpha - 1) \ln (A + W_1 \bar{h}) - \ln W_1} \right]
\]

\[
\geq \frac{-\varepsilon}{\sigma_\varepsilon} \geq \frac{X \beta + (\alpha - 1) \ln (1)}{-(\alpha - 1) \ln A - \ln W_1}
\]
Last term is:

\[
\frac{-1}{1 - \alpha} \left\{ \frac{1}{\sqrt{2\pi}} \left( e^{-\frac{C_1^2}{\sigma^2}} - e^{-\frac{C_2^2}{\sigma^2}} \right) \Phi \left( \frac{C_1}{\sigma\varepsilon} \right) \right. \\
\left. - \Phi \left( \frac{C_2}{\sigma\varepsilon} \right) \right\}
\]

\[
C_1 = \frac{X\beta + (\alpha - 1) \ln(1 - \bar{h})}{-\sigma\varepsilon} \\
C_2 = \frac{X\beta + (\alpha - 1)\ln(1) - (\alpha - 1)\ln A - \ln W_1}{\sigma\varepsilon}
\]
At $h = \bar{h}$ (with probability $P = P_3$) Q

$$E(h|h = \bar{h}) = \bar{h}P_3$$

$$P_3 = \Pr\left(\frac{1}{\sigma_\varepsilon} \left( \ln W_2 - X\beta - (\eta - 1) \ln (1 - \bar{h}) + (\alpha - 1) \ln (A + W_1 \bar{h}) \right) \leq \frac{\varepsilon}{\sigma_\varepsilon} \right)$$

etc.
Branch 2 labor supply

\[
\ln \left( \frac{W_2 h + (W_1 - W_2)h + A}{1 - h} \right) = \frac{\ln W_2}{1 - \alpha} - \frac{X \beta}{1 - \alpha} - \frac{\varepsilon}{1 - \alpha}
\]

\[
E \left( \frac{(W_1 - W_2)h + A}{1 - h} \mid h > \bar{h} \right) = \frac{\ln W_2}{1 - \alpha} - \frac{X \beta}{1 - \alpha}
\]

\[
- \frac{1}{1 - \alpha} E \left( \varepsilon \left| \frac{\ln W_2 + (\alpha - 1) \ln(A + W_1 \bar{h}) - X \beta - (\alpha - 1) \ln(1 - h)}{\sigma_\varepsilon} \right| \geq \frac{\varepsilon}{\sigma_\varepsilon} \right)
\]
Let

\[ Z^{(1)} = \ln \left( \frac{W_1 h + A}{1 - h} \right) \]

\[ Z^{(2)} = \ln \left( \frac{W_2 h + (W_1 - W_2)h + A}{1 - h} \right) \]

\[ E(Z^{(1)} | W_1, A, 0 < h < \bar{h}) = \frac{\ln W_1}{1 - \alpha} - \frac{X\beta}{1 - \alpha} - \frac{1}{1 - \alpha} E(\varepsilon | 0 < h < \bar{h}) \]

\[ E(Z^{(2)} | W_2, A, h > \bar{h}) = \frac{\ln W_2}{1 - \alpha} - \frac{X\beta}{1 - \alpha} - \frac{1}{1 - \alpha} E(\varepsilon | h > \bar{h}) \]

Can estimate by 2 stage methods.